# On the Solution of the Elliptic Interface Problems by Difference Potentials Method

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**Abstract** Designing numerical methods with high-order accuracy for problems in irregular domains and/or with interfaces is crucial for the accurate solution of many problems with physical and biological applications. The major challenge here is to design an efficient and accurate numerical method that can capture certain properties of analytical solutions in different domains/subdomains while handling arbitrary geometries and complex structures of the domains. Moreover, in general, any standard method (finite-difference, finite-element, etc.) will fail to produce accurate solutions to interface problems due to discontinuities in the model's parameters/solutions. In this work, we consider Difference Potentials Method (DPM) as an efficient and accurate solver for the variable coefficient elliptic interface problems.

## 1 Introduction

In this paper, we consider Difference Potentials Method (DPM) as an efficient and accurate solver for variable coefficient elliptic interface problems. DPM can be understood as the discrete version of the method of generalized Calderon's potentials and Calderon's boundary equations with projections in the theory of partial differential equations (PDEs). DPM introduces a computationally simple auxiliary domain. The original domain of the problem is embedded into an auxiliary domain, and the auxiliary domain is discretized using simple structured grids, e.g. Cartesian grids. After that, the main idea of DPM is to define a Difference Potentials operator, and to reformulate the original discretized PDEs (without imposed boundary/interface conditions yet) as equivalent discrete generalized Calderon's boundary equations with projections (BEP). These BEP are supplemented by the given boundary/interface

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conditions (the resulting BEP are always well-posed, as long as the original problem is well-posed), and solved to obtain the values of the solution at the points near the continuous boundary of the original domain (at the points of the discrete grid boundary which approximates the continuous boundary from the inside and outside of the domain). Using the obtained values of the solution at the discrete grid boundary, the approximation to the solution in the original domain is constructed through the discrete generalized Green's formula. *DPM offers geometric flexibility (without the use of unstructured meshes or "body-fitted" meshes), but does not require explicit knowledge of the fundamental solution, is not limited to constant coefficient problems, does not involve singular integrals, and can handle general boundary and/or interface conditions.* The reader can consult [18, 16] and [14, 15] for a detailed theoretical study of the methods based on Difference Potentials, and ([18, 16, 11, 12, 7, 20, 13, 19, 17, 3, 6, 5, 1], etc.) for the recent developments and applications of DPM.

In this paper, we extend the work on 1D and 2D elliptic interface problems started in [19, 17, 5] to variable coefficient elliptic interface models in 2D. A more detailed presentation of the methods with different high-order accurate discretizations, as well as the numerical analysis of DPM for the elliptic interface problems will be presented elsewhere [4].

The paper is organized as follows. In Section 2, we introduce the formulation of the problem. Next, in Section 2.1 we briefly describe the main building blocks of the DPM. Finally, we illustrate the performance of the proposed DPM, as well as compare DPM with the Mayo's method [10] and the Immersed Interface Method (IIM) [8, 9] in several challenging numerical experiments in Section 2.2.

## 2 Elliptic Interface Problem

In this work we consider interface/composite domain problem defined in some bounded domain $D^0 \subset \mathbb{R}^2$:

$$L_D u = \begin{cases} L_1 u_{D_1} = f_1(x,y) & (x,y) \in D_1 \\ L_2 u_{D_2} = f_2(x,y) & (x,y) \in D_2 \end{cases} \tag{1}$$

subject to the appropriate interface conditions:

$$u_{\overline{D}_1}\Big|_\Gamma - u_{\overline{D}_2}\Big|_\Gamma = \phi_1(x,y), \qquad \frac{\partial u_{\overline{D}_1}}{\partial n}\Big|_\Gamma - \frac{\partial u_{\overline{D}_2}}{\partial n}\Big|_\Gamma = \phi_2(x,y) \tag{2}$$

and boundary conditions

$$u|_{\partial D} = \psi(x,y) \tag{3}$$

where $D_1 \cup D_2 = D$ and $D \subset D^0$, see Fig. 1. Here, we assume $L_s$ , $s \in \{1,2\}$ are the second-order linear elliptic differential operators of the form

$$L_s u_{D_s} \equiv \frac{\partial}{\partial x}\Big(a_s(x,y)\frac{\partial u_{D_s}}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(b_s(x,y)\frac{\partial u_{D_s}}{\partial y}\Big), \quad s \in \{1,2\}.$$

The functions $a_s(x,y) \geq 1$ and $b_s(x,y) \geq 1$ are sufficiently smooth and defined in a larger auxiliary subdomains $D_s \subset D^0_s$.The functions $f_s(x,y)$ are sufficiently smooth functions defined in each subdomain $D_s$. We assume that the continuous problem (1)-(3) is well-posed. *Moreover, we assume that the operators $L_s$ are well-defined on some larger auxiliary domain $D^0_s$. More precisely, we assume that for any sufficiently smooth functions $f_s(x,y)$ the equations $L_s u_{D^0_s} = f_s(x,y)$ have a unique solution $u_{D^0_s}$ on $D^0_s$ that satisfy the given boundary conditions on $\partial D^0_s$.*

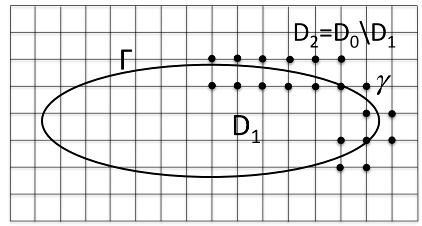


**Fig. 1** Example of an auxiliary domain $D^0$, original domains $D_1$ and $D_2$ separated by the interface $\Gamma$, and the example of the points in the discrete grid boundary set $\gamma$ for the 5-point stencil of the second-order method. Auxiliary domain $D^0$ coincides with $D$ here.

*Note, here and below, the upper/or lower index $s \in \{1,2\}$ is introduced to distinguish between the subdomains.*

## 2.1 Difference Potentials Method for Interface/Composite Domain Problems

Here we discuss the development of high-order methods based on Difference Potentials approach for the elliptic interface/composite domain problem (1)-(3). A more detailed presentation, as well as the numerical analysis of the method will be given elsewhere [4]. Also, the reader can consult [18], [19, 17, 5], [11], etc. for the detailed discussion and some examples of applications of DPM. Let us briefly describe the main steps of the algorithm.

**Introduction of the Auxiliary Domain:** Place the original domains $D_s$, $s \in \{1,2\}$ in the auxiliary computationally simple domains $D^0_s \subset \mathbb{R}^2$ that we will choose to be squares. Next, introduce a Cartesian mesh for each $D^0_s$, with points $x^s_j = j\Delta x^s, y^s_k = k\Delta y^s, (k, j = 0, \pm 1, ...)$. Let us assume for simplicity that $\Delta x^s = \Delta y^s := h^s$. Select discretization of the continuous model (1), for example here we will consider a finite-difference approximation. Next, define a finite-difference stencil $N^s_{j,k}$ with its center placed at $(x^s_j, y^s_k)$ (like a 5 node "dimension by dimension stencil" for the second-order scheme, or a 9 node "dimension by dimension stencil" for the classical fourth-order scheme, etc.). Additionally, introduce the point sets $M^0_s$ (the set of all the mesh nodes $(x^s_j, y^s_k)$ that belong to the interior of the auxiliary domain $D^0_s$), $M^+_s := M^0_s \cap D_s$ (the set of all the mesh nodes $(x^s_j, y^s_k)$ that belong to the interior of the original domain $D_s$), and by $M^-_s := M^0_s \backslash M^+_s$ (the set of all the mesh nodes $(x^s_j, y^s_k)$ that are inside of the auxiliary domain $D^0_s$ but don't belong to the interior of the original domain $D_s$). Define $N_s^+ := \{\bigcup_{j,k} N_{j,k}^s | (x_j^s, y_k^s) \in M_s^+\}$ (the set of all points covered by the stencil $N_{j,k}^s$ when center point $(x_j^s, y_k^s)$ of the stencil goes through all the points of the set $M_s^+ \subset D_s$). Similarly, define $N_s^- := \{\bigcup_{j,k} N_{j,k}^s | (x_j, y_k) \in M_s^-\}$ (the set of all points covered by the stencil $N_{j,k}^s$ when center point $(x_j^s, y_k^s)$ of the stencil goes through all the points of the set $M_s^-$).

Introduce $\gamma_s := N_s^+ \cap N_s^-$. The set $\gamma_s$ is called the *discrete grid boundary*. The mesh nodes from set $\gamma_s$ straddle the boundary $\partial D_s$. $N_s^0 := \{\bigcup_{j,k} N_{j,k}^s | (x_j^s, y_k^s) \in M_s^0\} \subset \overline{D_s^0}$. The sets $N_s^0$, $M_s^0$, $N_s^+$, $N_s^-$, $M_s^+$, $M_s^-$, $\gamma_s$ will be used to develop the method based on the Difference Potentials approach, Fig. 1.

**Difference Equations:** The discrete reformulation of the model problem (1) in each auxiliary domain $D_s^0$ is: solve for $u_{j,k}^s \in N_s^+$

$$L_h^s[u_{j,k}^s] = F_{j,k}^s, \quad (x_j^s, y_k^s) \in M_s^+ \tag{4}$$

where $L_h^s[u_{j,k}^s]$ is the discrete linear elliptic operator obtained using finite-difference approximation of order $r$ (for example, the second-order $r=2$ or the fourth-order $r=4$, etc.). $F_{j,k}^s$ denotes the discrete right-hand side. $u_{j,k}^s :\approx u_{D_s}(x_j^s, y_k^s)$. The unknowns are $u_{j,k}^s :\approx u_{D_s}(x_j^s, y_k^s)$, where $(x_j^s, y_k^s)$ is a mesh point of the Cartesian grid.

We need to complete the linear system of difference equations (4) with the appropriate choice of the numerical boundary and interface conditions to construct a unique accurate approximation of the continuous problem (1)-(3) in domain $D$. Thus, to design an efficient algorithm for any type of boundary and interface conditions, we will consider a numerical method based on the idea of the Difference Potentials.

*Step 1: Construction of a Particular Solution:* Denote by $u_{j,k}^s := G_s^h F_{j,k}^s, \quad u_{j,k}^s \in N_s^+$ the particular solution of the discrete problem (4), which we will construct as the solution (restricted to set $N_s^+$) of the simple auxiliary problem (AP) of the following form:

$$L_h^s[u_{j,k}^s] = \begin{cases} F_{j,k}^s, & (x_j^s, y_k^s) \in M_s^+, \\ 0, & (x_j^s, y_k^s) \in M_s^-, \end{cases} \tag{5}$$

$$u_{j,k}^s = 0, \quad (x_j^s, y_k^s) \in N_s^0 \backslash M_s^0 \tag{6}$$

*Step 2: Difference Potentials and Construction of the BEP:* We now introduce a linear space $\mathbf{V}_{\gamma_s}$ of all the grid functions denoted by $v_{\gamma_s}$ defined on $\gamma_s$ [18], [19, 17, 5], etc. We will extend the value $v_{\gamma_s}$ by zero to other points of the grid $N_s^0$.

**Definition 1.** The Difference Potential with any given density $v_{\gamma_s} \in \mathbf{V}_{\gamma_s}$ is the grid function $u_{j,k}^s := \mathbf{P}_{N^+\gamma_s} v_{\gamma_s}$, defined on $N_s^+$, and coincides on $N_s^+$ with the solution $u_{j,k}^s$ of the simple auxiliary problem (AP) of the following form:

$$L_h^s[u_{j,k}^s] = \begin{cases} 0, & (x_j^s, y_k^s) \in M_s^+, \\ L_h^s[v_{\gamma_s}], & (x_j^s, y_k^s) \in M_s^-, \end{cases} \tag{7}$$

$$u_{j,k}^s = 0, \quad (x_j^s, y_k^s) \in N_s^0 \backslash M_s^0 \tag{8}$$

Here, $\mathbf{P}_{N^+\gamma_s}$ denotes the operator which constructs the Difference Potential $u^s_{j,k} = \mathbf{P}_{N^+\gamma_s} v_{\gamma_s}$ from the given density $v_{\gamma_s} \in V_{\gamma_s}$. The operator $\mathbf{P}_{N^+\gamma_s}$ is the linear operator of the density $v_{\gamma_s}$. Hence, it can be easily constructed [19, 17, 5]. We will now state the most important theorem of the method:

**Theorem 1.** *Density $u_{\gamma_s}$ is the trace of some solution $u^s_{j,k} \in N^+_s$ to the Difference Equations (4) : $u_{\gamma_s} \equiv Tr_{\gamma_s} u^s_{j,k}$, if and only if, $u_{\gamma_s}$ satisfies Generalized Calderon's Boundary Equations with Projections (BEP)*

$$u_{\gamma_s} - \mathbf{P}_{\gamma_s} u_{\gamma_s} = G^h_s F_{\gamma_s}, \tag{9}$$

*where $G^h_s F_{\gamma_s} := Tr_{\gamma_s}(G^h_s F^s_{j,k})$ is the trace (or restriction) of the particular solution $G^h_s F^s_{j,k} \in N^+_s$ constructed in (5)-(6) on the grid boundary $\gamma_s$, and $\mathbf{P}_{\gamma_s} u_{\gamma_s} := Tr_{\gamma_s}(\mathbf{P}_{N^+\gamma_s} u_{\gamma_s})$ is the trace of the Difference Potential $\mathbf{P}_{N^+\gamma_s} u_{\gamma_s} \in N^+_s$ in (7)-(8) on the grid boundary $\gamma_s$.*

*Remark:* The BEP (9) are constructed for each subdomain and solved efficiently together with the boundary and interface conditions for the unknown densities $u_{\gamma_s}$ using the idea of the extension operator for $u_{\gamma_s}$, and the spectral approach for the approximation of the Cauchy data $(u^s, \frac{\partial u^s}{\partial n})|_{\partial D_s}$ ([19, 17, 11], etc.).

*Step 3: Construction of the Approximate Solution to the Model Problem (1)-(3) from the density $u_{\gamma_s}$ obtained in Step 2:*

**Statement 1** *(Generalized Green's Formula)*
*The discrete solution $u^s_{j,k} := \mathbf{P}_{N^+\gamma_s} u_{\gamma_s} + G^h_s F^s_{j,k}$ is the approximation to the solution $u^s_{j,k} \approx u^s(x^s_j, y^s_k)$, $(x^s_j, y^s_k) \in N^+_s \cap D_s$ of the continuous problem (1)-(3).*

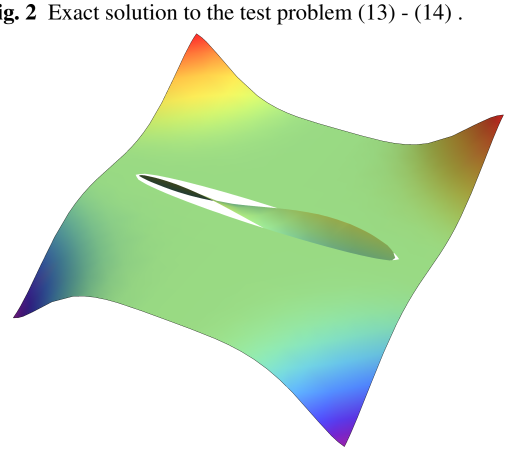
*The expected accuracy of the proposed method for domains with the smooth boundaries and under sufficient regularity of the exact solutions will be $O(h^{r-\varepsilon})$ in the discrete Hölder norm of order $2+\varepsilon$ (if the continuous second-order linear elliptic operator L is approximated with $r^{th}$ order of accuracy by the discrete operator $L_h$), [15, 14, 18], [4] and Section 2.2. Here, $0 < \varepsilon < 1$ is an arbitrary small number.*

## 2.2 Numerical Examples

In the numerical examples below, we consider a second-order centered finite-difference approximation (with 5-node stencil) as the underlying discretization for DPM. The numerical experiments for the fourth-order approximation will be presented elsewhere [4].

The first test problem that we consider here is the prob-

**Fig. 2** Exact solution to the test problem (13) - (14) .



lem from the paper [2]:

$$\Delta u_{D_s} = f_s(x,y), \quad (x,y) \in D_s, \quad s \in \{1,2\} \tag{10}$$

where the interface between two subdomains $D_1$ and $D_2$ (see Fig. 1) is given by an ellipse with semi-axes $(a,b) = (0.9,0.1)$, and the curvature is $\kappa = -90$ at $(\pm a, 0)$ which leads to a quite challenging test [2]. The exact solution here is

$$u_1 = \sin x \cos y, \quad u_2 = 0, \tag{11}$$

which is discontinuous at the interface. The results for the test problem (10)-(11) are presented in Table 1, which shows the relative error in the maximum norm of the solution and its derivatives. To match the settings of the numerical experiments in paper [2], we consider auxiliary domains (here and below) $D_1^0 = D_2^0 \equiv D = [-1.1, 1.1] \times [-1.1, 1.1]$ for the subdomains $D_1$ and $D_2$ respectively, Fig. 1. Note, that in these settings, $h^1 = h^2 = h$ (however, DPM handles as easily different auxiliary problems/non-matching meshes [19, 17, 6, 5, 1]). As observed from the Table 1 here, and from the Table 1 (bottom), on page 111 in paper [2], the accuracy in the solution for the test problem (10)-(11) obtained by DPM is very close to the accuracy obtained by Mayo's Method and by IIM. But, the accuracy in the derivatives of the solution obtained by DPM is superior to the accuracy obtained by Mayo's Method or IIM.

**Table 1** Test problem (10) - (11) with $a = 0.9$, $b = 0.1$ from paper [2]. Here $h = 0.055$ and $h = 0.0034$ corresponds to $N = 20$ and $N = 320$ (half of the number of the subintervals) respectively in Table 1 (bottom), page 111 in [2]. Relative $L_\infty$ error in the solution and in its derivatives.

| $h$ | $L_\infty$-error in $u$ | Rate | $L_\infty$-error in $u_x$ | Rate | $L_\infty$-error in $u_y$ | Rate |
|---|---|---|---|---|---|---|
| 0.055 | $3.4368e-05$ | | $2.6399e-05$ | | $8.9673e-05$ | |
| 0.0275 | $7.2220e-06$ | 2.25 | $1.0770e-05$ | 1.29 | $1.0713e-05$ | 3.20 |
| 0.0138 | $1.4932e-06$ | 2.27 | $1.6495e-06$ | 2.71 | $1.5316e-06$ | 2.81 |
| 0.0069 | $5.5244e-07$ | 1.43 | $5.5440e-07$ | 1.57 | $7.0618e-07$ | 1.12 |
| 0.0034 | $8.3612e-08$ | 2.72 | $6.2907e-08$ | 3.14 | $7.7577e-08$ | 3.19 |
| 0.0017 | $2.0790e-08$ | 2.01 | $3.4931e-09$ | 4.17 | $2.9627e-09$ | 4.71 |

The second test problem is again from [2] and has the same settings as the first test problem (10)-(11), but now the exact solution is defined as:

$$u_1 = x^9 y^8, \quad u_2 = 0. \tag{12}$$

The results for this test problem are presented in Table 2. Compared to the DPM results of the previous test problem (10)-(11), the DPM errors for the test problem (10), (12) are much smaller than the errors for Mayo's method and IIM, reported in Table 3, page 113 in [2].

As the last and more challenging test problem, we consider the interface problem with variable coefficients as described below:

**Table 2** Test problem (10), (12) with $a = 0.9$, $b = 0.1$ from paper [2]. Here $h = 0.0275$ and $h = 0.0017$ corresponds to $N = 40$ and $N = 640$ (half of the number of the subintervals) respectively in Table 3, page 113 in [2]. Relative $L_\infty$ error in the solution and its derivatives.

| $h$ | $L_\infty$-error in $u$ | Rate | $L_\infty$-error in $u_x$ | Rate | $L_\infty$-error in $u_y$ | Rate |
|---|---|---|---|---|---|---|
| 0.0275 | $3.4725e-04$ | | $3.2322e-04$ | | $3.1602e-04$ | |
| 0.0138 | $8.6281e-05$ | 1.85 | $4.4379e-05$ | 2.31 | $4.3989e-05$ | 2.84 |
| 0.0069 | $2.1469e-05$ | 1.93 | $5.8286e-06$ | 2.40 | $5.8162e-06$ | 2.92 |
| 0.0034 | $5.3542e-06$ | 1.96 | $7.4737e-07$ | 2.45 | $7.4825e-07$ | 2.96 |
| 0.0017 | $1.3366e-06$ | 1.98 | $9.4637e-08$ | 2.48 | $9.4906e-08$ | 2.98 |

$$\frac{\partial}{\partial x}\left(a_s(x,y)\frac{\partial u_{D_s}}{\partial x}\right)+\frac{\partial}{\partial y}\left(b_s(x,y)\frac{\partial u_{D_s}}{\partial y}\right)=f_s(x,y),\quad (x,y)\in D_s,\quad s\in\{1,2\} \tag{13}$$

where $a_1 = b_1 = 10^6$, $a_2 = (3+0.5\sin(2x+y))$ and $b_2 = (2+0.5\cos(4x+3y))$. The interface curve for this problem is again given by the ellipse with semi-axes $(a,b) = (0.9,0.1)$. The exact solution for this test problem (13) is set to

$$u_1 = \sin(y^2x)\sin(x^3y), \quad u_2 = \sin(2x)\sin(3y). \tag{14}$$

The interface problem (13)-(14) is much more challenging than the previous test problems since it has discontinuous solution at the interface, as well as a large jump ratio between diffusion coefficients in subdomains $D_1$ and $D_2$, Fig. 2. The results for this test problem are presented in Table 3, which shows the relative error of the solution and its derivatives in the maximum norm. As in the previous numerical examples, DPM preserves overall second-order (and even slightly better in the derivative) accuracy in the solution and its derivatives.

**Table 3** Test problem (13) - (14) with $a = 0.9$, $b = 0.1$. Relative $L_\infty$ error in the solution and its derivatives.

| $h$ | $L_\infty$-error in $u$ | Rate | $L_\infty$-error in $u_x$ | Rate | $L_\infty$-error in $u_y$ | Rate |
|---|---|---|---|---|---|---|
| 0.0275 | $4.5671e-04$ | | $1.3639e-04$ | | $1.3981e-03$ | |
| 0.0138 | $1.1520e-04$ | 1.99 | $2.2087e-05$ | 2.63 | $3.1356e-04$ | 2.16 |
| 0.0069 | $2.8329e-05$ | 2.02 | $2.3138e-06$ | 3.25 | $3.5176e-05$ | 3.16 |
| 0.0034 | $7.0319e-06$ | 2.01 | $3.1931e-07$ | 2.86 | $4.6670e-06$ | 2.91 |
| 0.0017 | $1.7578e-06$ | 2.00 | $4.9421e-08$ | 2.69 | $7.2111e-07$ | 2.69 |

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