# On surface radiation conditions for an ellipse 

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#### Abstract

We compare several On Surface Radiation Boundary Conditions in two dimensions, for solving the Helmholtz equation exterior to an ellipse. We also introduce a new boundary condition for an ellipse based on a modal expansion in Mathieu functions. We compare the OSRC to a finite difference method.


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## 1. Introduction

We consider scattering exterior to a two-dimensional body based on the Helmholtz equation that describes wave motion in frequency space

$$
\begin{equation*}
\Delta u+k^{2} u=0 \tag{1}
\end{equation*}
$$

For a numerical solution one needs to truncate the unbounded domain and introduce an artificial surface with a radiation boundary condition. Kriegsmann et al. [1] instead proposed imposing these boundary conditions directly on the scatterer. This yields both the function and its normal derivative on the scatterer from which the solution at all points can be calculated.

Bayliss and Turkel [2] and later Gunzburger (BGT) [3] constructed a sequence of radiation boundary conditions based on matching terms in an expansion, in polar coordinates, in the inverse radius $\frac{1}{R}$. The most popular is BGT2 which contains a first order normal derivative and a second order tangential derivative. Many attempts have been made to construct conditions for more general shapes. Remarkably, most of these approaches reduced to the BGT condition for a circle, at least through second order. However, they differ in the boundary condition constructed for other shapes. We consider scattering about an ellipse for which the exact solution is known. We compare several approaches for use as an On Surface Radiation Condition (OSRC). The boundary conditions we compare are those of [4,5,1,6-9].

## 2. Radiation boundary conditions

One approach to construct an artificial boundary condition is to match an infinite series to the solution where the functional form of the terms is known. The boundary condition is constructed to match the first terms of the outgoing solution. Bayliss and Turkel $[2,3]$ constructed a sequence of such boundary conditions, in a recursive manner, that are more accurate as $R \rightarrow \infty$. One then uses the interior equation to eliminate radial derivatives beyond the first in terms of tangential derivatives. The most popular boundary condition has been BGT2 since this involves only second tangential derivatives. For scattering about a circle an alternative is a modal expansion in Hankel functions [4,10]. For $k R$ large this gives results similar to the BGT approach. However, for small wavenumbers it is significantly better [11].

When the outer surface is not a circle there are several approaches to generalize the BGT boundary conditions. When the surface is given analytically one can rederive the formulae based on an expansion for the solution of the Helmholtz

[^0]equation in terms of coordinates that give the outer surface. Another approach is to derive a global formula which couples all the points on the boundary. One then approximates the integral to get a local boundary condition. This again gives the BGT formulae, for a circle, in terms of the normal and tangential directions [4,12,13]. If the outer surface is an ellipse one can convert ( $r, \theta$ ) derivatives to elliptical coordinates or alternatively derive an expansion in terms of elliptical coordinates and match this expansion. For the infinite series the two approaches are the same. However, using only a finite number of terms (usually two) the two approaches differ. In the first approach the assumption is that the solution is well represented by a few circular waves and this is transformed to elliptical coordinates while in the second approach the solution is assumed to be well approximated by the first two elliptical waves. This is true, in particular, for an OSRC imposed directly on the scatterer. Another approach is an expansion in terms of an inverse wavenumber. For scattering about an ellipse we will construct a new modal expansion in Mathieu functions. More details of the various approaches are given in [14].

### 2.1. Radiation boundary conditions

In two dimensions, the solution to the exterior Helmholtz equation has a convergent expansion

$$
\begin{equation*}
u(r, \theta)=H_{0}(k r) \sum_{j=0}^{\infty} \frac{F_{j}(\theta)}{(k r)^{j}}+H_{1}(k r) \sum_{j=0}^{\infty} \frac{G_{j}(\theta)}{(k r)^{j}} \tag{2}
\end{equation*}
$$

Instead Bayliss et al. worked with the asymptotic expansion

$$
\begin{equation*}
u(r, \theta) \sim \sqrt{\frac{2}{\pi k r}} \mathrm{e}^{i\left(k r-\frac{\pi}{2}\right)} \sum_{j=0}^{\infty} \frac{f_{j}(\theta)}{(k r)^{j}} . \tag{3}
\end{equation*}
$$

Matching the two-dimensional asymptotic expansion (3) through two terms yields

$$
\left(\frac{\partial}{\partial r}-\mathrm{i} k+\frac{5}{2 r}\right)\left(\frac{\partial}{\partial r}-\mathrm{i} k+\frac{1}{2 r}\right) u=0
$$

or

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial r^{2}}+\left(\frac{3}{r}-2 \mathrm{i} k\right) \frac{\partial u}{\partial r}-\left(k^{2}+\frac{3 \mathrm{i} k}{r}-\frac{3}{4 r^{2}}\right) u=0 \tag{4}
\end{equation*}
$$

We now use the Helmholtz equation in polar coordinates

$$
\frac{\partial^{2} u}{\partial r^{2}}+\frac{1}{r} \frac{\partial u}{\partial r}+k^{2} u+\frac{1}{r^{2}} \frac{\partial^{2} u}{\partial \theta^{2}}=0
$$

to eliminate $\frac{\partial^{2} u}{\partial r^{2}}$. This yields BGT2

$$
\begin{equation*}
\frac{\partial u}{\partial r}=\mathrm{i} k u-\frac{u}{2 r}+\frac{u}{8 r^{2}\left(\frac{1}{r}-\mathrm{i} k\right)}+\frac{1}{2 r^{2}\left(\frac{1}{r}-\mathrm{i} k\right)} \frac{\partial^{2} u}{\partial \theta^{2}} \tag{5}
\end{equation*}
$$

Using instead the full convergent expansion (2) and $H_{0}^{\prime}(z)=-H_{1}(z)$ and $H_{1}^{\prime}(z)=H_{0}(z)-\frac{1}{z} H_{1}(z)$ one gets [12,13,10,11]

$$
\begin{equation*}
\frac{\partial u}{\partial r}=-k\left[\frac{H_{1}(k r)}{H_{0}(k r)} u+\left(\frac{H_{1}(k r)}{H_{0}(k r)}+\frac{H_{0}(k r)}{H_{1}(k r)}-\frac{1}{k r}\right) \frac{\partial^{2} u}{\partial \theta^{2}}\right] . \tag{6}
\end{equation*}
$$

Using the asymptotic properties of the Hankel functions for large argument the modal expansion agrees with BGT2 through $O\left(\frac{1}{(k r)^{2}}\right)$ for large $k r$. For high frequencies it is more convenient to use (5). For low frequencies it is necessary to use (6), see [11] for more details.

An ellipse, $\xi=\xi_{0}$, with semi-major and semi-minor axes $a$ and $b$ respectively is given parametrically by

$$
\begin{equation*}
x=a \cos (\eta)=f \cosh \left(\xi_{0}\right) \cos (\eta) \quad y=b \sin (\eta)=f \sinh \left(\xi_{0}\right) \sin (\eta) \tag{7}
\end{equation*}
$$

Defining $h_{\xi}=h_{\eta}=\frac{\partial s}{\partial \eta}$, we have

$$
h_{\xi}=f \sqrt{\cosh ^{2}(\xi)-\cos ^{2}(\eta)} \xrightarrow{\text { on ellipse }} \sqrt{a^{2}-f^{2} \cos ^{2}(\eta)}=\sqrt{a^{2} \sin ^{2}(\eta)+b^{2} \cos ^{2}(\eta)}
$$

and

$$
\frac{\partial u}{\partial n}=\frac{1}{h_{\xi}} \frac{\partial u}{\partial \xi} \quad \frac{\partial u}{\partial s}=\frac{1}{h_{\xi}} \frac{\partial u}{\partial \eta}
$$

Several of the OSRC are given in terms of $\frac{\partial u}{\partial \eta}$ while others are in terms of $\frac{\partial u}{\partial n}$. The Helmholtz equation, in elliptical coordinates, is given by

$$
\frac{\partial^{2} u}{\partial \xi^{2}}+\frac{\partial^{2} u}{\partial \eta^{2}}+h_{\xi}^{2} k^{2} u=0
$$

and the curvature of the ellipse is given by

$$
\begin{equation*}
\kappa=\frac{a b}{h_{\xi}^{3}} . \tag{8}
\end{equation*}
$$

Grote and Keller [4] and later Thompson, Huan and Ianculescu [15] found that for an ellipse, using an expansion in Mathieu functions, coupled with a DtN formula the BGT2 condition remains with $r$ replaced by the ellipse radius $f \cosh (\xi)$. It yields at the ellipse $\xi=\xi_{0}$

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial \xi^{2}}+\alpha \frac{\partial u}{\partial \xi}+\beta u=0 \tag{9}
\end{equation*}
$$

with (using $\left.a=f \cosh \left(\xi_{0}\right), b=f \sinh \left(\xi_{0}\right)\right)$

$$
\begin{aligned}
& \alpha=3 \tanh \left(\xi_{0}\right)-2 \mathrm{i} k f \sinh \left(\xi_{0}\right)-\operatorname{coth}\left(\xi_{0}\right)=3 \frac{b}{a}-2 \mathrm{i} k b-\frac{a}{b} \\
& \beta=\frac{3}{4} \tanh ^{2}\left(\xi_{0}\right)-\left(k f \sinh \left(\xi_{0}\right)\right)^{2}-3 \mathrm{i} k f \sinh \left(\xi_{0}\right) \tanh \left(\xi_{0}\right)=\frac{3}{4} \frac{b^{2}}{a^{2}}-k^{2} b^{2}-3 \mathrm{i} k \frac{b^{2}}{a} .
\end{aligned}
$$

They [15] converted from $\xi$ derivatives to normal derivatives and used the Helmholtz equation in elliptical coordinates to eliminate the $\frac{\partial^{2} u}{\partial \xi^{2}}$ derivative. This gives

$$
\begin{equation*}
h_{\xi} \frac{\partial u}{\partial n}=\frac{\partial u}{\partial \xi}=\frac{1}{\alpha}\left(\left(k^{2} h_{\xi}^{2}-\beta\right) u+\frac{\partial^{2} u}{\partial \eta^{2}}\right) . \tag{10}
\end{equation*}
$$

Reiner, Djellouli and Harari [5] considered an OSRC for an ellipse based on the condition of Grote and Keller [4]. They used the Helmholtz equation in polar coordinates to eliminate the second radial derivative. Given an ellipse with semi-axes $a$ and $b$ they derived

$$
\begin{equation*}
\frac{\partial u}{\partial \xi}=\frac{b}{a}\left[\left(\mathrm{i} k a-\frac{1}{2}+\frac{1}{8(1-\mathrm{i} k a)}\right) u+\frac{1}{2(1-\mathrm{i} k a)} \frac{\partial^{2} u}{\partial \eta^{2}}\right] . \tag{11}
\end{equation*}
$$

The first to consider general shapes for the outer boundary was Kriegsmann, Taflove and Umashankar [1]. They considered the BGT formulae for the circle (5) and formally replaced $\frac{\partial}{\partial r} \rightarrow \frac{\partial}{\partial n}, \frac{1}{r} \rightarrow \kappa$ where $\kappa$ is the curvature and $\frac{1}{r^{2}} \frac{\partial^{2} u}{\partial \theta^{2}} \rightarrow \frac{\partial^{2} u}{\partial s^{2}}$. They arrived at the boundary condition

$$
\begin{equation*}
\frac{\partial u}{\partial n}=\mathrm{i} k u-\frac{\kappa u}{2}-\frac{\kappa^{2} u}{8(\mathrm{i} k-\kappa)}-\frac{1}{2(\mathrm{i} k-\kappa)} \frac{\partial^{2} u}{\partial s^{2}} \tag{12}
\end{equation*}
$$

Later Jones $[16,6]$ suggested including derivatives of the curvature:

$$
\begin{equation*}
\frac{\partial u}{\partial n}=\mathrm{i} k u-\frac{\kappa u}{2}-\frac{\kappa^{2} u}{8(\mathrm{i} k-\kappa)}-\frac{1}{2(\mathrm{i} k-\kappa)} \frac{\partial^{2} u}{\partial s^{2}}+\frac{\mathrm{i} k}{\mathrm{i} k-\kappa}\left[\frac{1}{8 k^{2}} \frac{\mathrm{~d}^{2} \kappa}{\mathrm{ds}^{2}} u+\frac{1}{2 k^{2}} \frac{\partial \kappa}{\partial s} \frac{\partial u}{\partial s}\right] . \tag{13}
\end{equation*}
$$

Antoine [9] has reformulated these schemes in a symmetric manner. This seems to be of importance for a finite element implementation.

Antoine, Barucq and Bendali [8] considered a decomposition into incoming and outgoing wave constructed, based on pseudo-differential operators, an asymptotic expansion in $\frac{1}{k}$ for general bodies. For the two terms that they calculated, they recovered the BGT boundary condition for the circle. For general two-dimensional shapes they derived a full second order operator

$$
\begin{equation*}
\frac{\partial u}{\partial n}=\mathrm{i} k u-\frac{\kappa u}{2}-\frac{\kappa^{2} u}{8(\mathrm{i} k-\kappa)}+\frac{1}{8 \mathrm{k}^{2}} \frac{\mathrm{~d}^{2} \kappa}{\mathrm{ds}^{2}} u-\frac{\partial}{\partial s}\left(\frac{1}{2(\mathrm{i} k-\kappa)} \frac{\partial u}{\partial s}\right) . \tag{14}
\end{equation*}
$$

Kallivokas et al. [7] considered a geometric optics type expansion and developed a second order absorbing boundary condition

$$
\begin{align*}
\frac{\partial u}{\partial n} & =\mathrm{i} k u-\frac{\kappa u}{2}-\frac{1}{2(\mathrm{i} k-\kappa)}\left(\frac{\partial^{2} u}{\partial s^{2}}+\frac{1}{4} \kappa^{2} u\right)-\frac{1}{2(\mathrm{i} k-\kappa)^{2}}\left[\frac{1}{4} \frac{\mathrm{~d}^{2} \kappa}{\mathrm{~d} s^{2}} u+\frac{\mathrm{d} \kappa}{\mathrm{ds}} \frac{\partial u}{\partial s}\right] \\
& =\left(\mathrm{i} k-\frac{\kappa}{2}-\frac{\kappa^{2}}{8(\mathrm{i} k-\kappa)}-\frac{1}{8(\mathrm{i} k-\kappa)^{2}} \frac{\mathrm{~d}^{2} \kappa}{\mathrm{~d} s^{2}}\right) u-\frac{\partial}{\partial s}\left(\frac{1}{2(\mathrm{i} k-\kappa)} \frac{\partial u}{\partial s}\right) . \tag{15}
\end{align*}
$$

The above formulae are all symmetric, in the sense of $L_{2}$, except for those of Kriegsmann et al. (12) and Jones (13). Antoine [9] suggested symmetrizing these formulae by bringing the coefficients inside the derivative or equivalently eliminating the first derivative terms. This mainly effects finite element methods and is discussed in more detail in [17]. The difference between the symmetric versions is the coefficient of the term $\frac{\mathrm{d}^{2} \kappa}{\mathrm{ds}}{ }^{2} u$. Hence, for large $k$ (relative to the curvature) these methods are similar. For small $k$ the boundary condition of Antoine et al. becomes infinite and less accurate.

### 2.2. New absorbing boundary condition

We recall the elliptical coordinates (7). The Helmholtz equation is given by

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial \xi^{2}}+\frac{\partial^{2} u}{\partial \eta^{2}}+\frac{k^{2} f^{2}}{2}(\cosh (2 \xi)-\cos (2 \eta)) u=0 \tag{16}
\end{equation*}
$$

Assume $u=\psi(\xi) \lambda(\eta)$. Then $\frac{\mathrm{d}^{2} \psi}{\mathrm{~d} \xi^{2}}+\left(\frac{k^{2} f^{2}}{2} \cosh (2 \xi)-l\right) \psi=0$. $l$ is separation constant determined so that $\lambda(\eta)$ is periodic which leads to either even or odd solutions with eigenvalues $l_{r}(q)$ for $r$ even, where $q$ is

$$
\begin{equation*}
q=\frac{k^{2} f^{2}}{4}=\frac{k^{2}}{4}\left(a^{2}-b^{2}\right) \tag{17}
\end{equation*}
$$

We choose $\psi_{j}, j=1$, 2 as the first two even Mathieu-Hankel functions, $M_{0}(\xi), M_{1}(\xi)$ with the corresponding characteristic values $l_{0}, l_{1}$. We assume an expansion in Mathieu functions (cf. [10])

$$
\begin{align*}
& u(\xi, \eta)=\lambda_{0}(\eta) M_{0}(\xi)+\lambda_{1}(\eta) M_{1}(\xi)  \tag{18a}\\
& \frac{\partial u}{\partial \xi}(\xi, \eta)=\lambda_{0}(\eta) M_{0}^{\prime}(\xi)+\lambda_{1}(\eta) M_{1}^{\prime}(\xi)  \tag{18b}\\
& \frac{\partial^{2} u}{\partial \xi^{2}}(\xi, \eta)=\lambda_{0}(\eta) M_{0}^{\prime \prime}(\xi)+\lambda_{1}(\eta) M_{1}^{\prime \prime}(\xi) . \tag{18c}
\end{align*}
$$

Solving for $\lambda_{0}$ and $\lambda_{1}$ at $\xi=\xi_{0}$, from the first two equations we get

$$
\lambda_{0}=\left.\frac{M_{1}^{\prime} u-M_{1} \frac{\partial u}{\partial \xi}}{\left(M_{0} M_{1}^{\prime}-M_{0}^{\prime} M_{1}\right)}\right|_{\xi=\xi_{0}} \quad \lambda_{1}=-\left.\frac{M_{0}^{\prime} u-M_{0} \frac{\partial u}{\partial \xi}}{\left(M_{0} M_{1}^{\prime}-M_{0}^{\prime} M_{1}\right)}\right|_{\xi=\xi_{0}}
$$

This exists and is unique when $W\left(M_{0}, M_{1}\right)=M_{0} M_{1}^{\prime}-M_{0}^{\prime} M_{1} \neq 0$. The Wronskian $W\left(M_{0}, M_{1}\right)$ is nonzero when the functions are linearly independent. Since the Mathieu functions are the solution of a Sturm-Liouville problem they are linearly independent. Substituting in (18c) one gets

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial \xi^{2}}+\frac{M_{1} M_{0}^{\prime \prime}-M_{0} M_{1}^{\prime \prime}}{M_{0} M_{1}^{\prime}-M_{0}^{\prime} M_{1}} \frac{\partial u}{\partial \xi}+\frac{\left(-M_{1}^{\prime} M_{0}^{\prime \prime}+M_{0}^{\prime} M_{1}^{\prime \prime}\right)}{M_{0} M_{1}^{\prime}-M_{0}^{\prime} M_{1}} u=0 \tag{19}
\end{equation*}
$$

Define

$$
T=\left.\frac{\frac{M_{0}^{\prime}}{M_{0}}-\frac{M_{1}^{\prime}}{M_{1}}}{l_{1}-l_{0}}\right|_{\xi=\xi_{0}}
$$

We use the Mathieu equation for $M_{0}^{\prime \prime}$ and $M_{1}^{\prime \prime}$ to simplify (19). We use the Helmholtz equation to eliminate $\frac{\partial^{2} u}{\partial \xi^{2}}$ in terms of $\frac{\partial^{2} u}{\partial \eta^{2}}$. This yields

$$
\begin{equation*}
\frac{\partial u}{\partial \xi}=T \frac{\partial^{2} u}{\partial \eta^{2}}+\left(T\left(l_{0}-2 q \cos (2 \eta)\right)+\frac{M_{0}^{\prime}}{M_{0}}\right) u \tag{20}
\end{equation*}
$$

This boundary condition is exact for the first two even modes. The boundary condition using Hankel functions (6) reduces to the BGT2 condition (5) for large ka. The elliptical coordinate system is singular in the limit of equal semi-major and semiminor axes. Hence, it is difficult to prove that we recover the BGT formula. Computational tests show that the new boundary condition behaves nicely as the aspect ratio of the ellipse approaches 1 . However, it becomes difficult to computationally evaluate the Mathieu functions, in the exact solution, for aspect ratios near 1. An anonymous reviewer pointed out that this separation of variables can be viewed as a diagonalization of the Helmholtz operator in harmonics over the ellipse. For a slightly different approach see [18,19].

## 3. Results

We consider several radiation boundary conditions imposed directly on a scattering ellipse (OSRC) [1]. Curvature terms on the ellipse are calculated analytically (8). The physical boundary condition is either a Dirichlet (soft scatterer) or Neumann (hard scatterer) condition on the body based on a plane wave impingement. The OSRC gives either the normal derivative (soft scatterer) or the function (hard scatterer) on the body which we compare to the exact one. For the hard scatterer this results in an ordinary second order differential equation in the tangential direction. We solve this by second order accurate finite difference scheme. Both symmetric and nonsymmetric OSRCs are straightforward. In all cases the angle of the incident plane wave is 0 , the major axis is 1 and we vary the minor axis. Both the exact solution and the numerical approximation are normalized by their respective norms before computing the error.


Fig. 1. Dependence on parameters.
We display the dependence of the OSRC boundary conditions on various parameters. We first consider a Neumann boundary and chose five boundary conditions to compare. In Fig. 1(a) we show the dependence of the wavenumber for an aspect 2 ellipse and zero angle of incidence. In Fig. 1(b) we set $k=1$ and a zero angle and show the dependence on the aspect ratio. In Fig. 2, $k=1$, the aspect ratio equals 2 and we show the dependence on the angle of incidence. We find that the accuracy is fairly independent of the angle. In all cases the new modal condition is best followed by that of Reiner et al. The boundary conditions of Antoine et al. (14) and Kallikovas et al. (15) are not shown in the figures to make them more readable. They are presented in the tables.

The OSRCs can also be implemented as an absorbing boundary condition on an outer artificial boundary for a finite difference or finite element method. We next investigate the accuracy of the OSRC by comparing it with a finite difference (FD) solution. The FD is solved, for the scattered wave, in elliptical coordinates in a domain bounded by the inner elliptical scatterer with a Dirichlet or Neumann boundary condition given by the impinging plane wave and a concentric exterior ellipse with an absorbing boundary condition of the type discussed above. The conditions of Kriegsmann et al. and Jones can be modified to be symmetric [9]. In this study they were implemented in their original nonsymmetric form. Further details and results are given in [17] which implements the nonsymmetric form for a finite difference scheme and the symmetric form for a finite element scheme. Curvature terms on the outer ellipse are calculated analytically (8).

We check the effect of the semi-major axis of the outer ellipse ( $a=1.1$ and $a=1.5$ ) on the accuracy. In Fig. 3 we compare several methods with the exact solution for $k=1$ and an aspect ratio of 2 . The error between the approximate


Fig. 2. Dependence on angle of incidence.


Fig. 3. Dirichlet $\mathrm{bc}, k=1, \mathrm{AR}=2$.


Fig. 4. Dirichlet $\mathrm{bc}, k=1, \mathrm{AR}=10$.
OSRC solutions and the exact normal derivative is given in the legend. In Fig. 4 we consider $k=1$ but aspect ratio 10. In Fig. 5 we consider an aspect ratio of 2 but with $k=4$. In all the cases the OSRC solution at $180^{\circ}$ (i.e. behind the ellipse in the shadow region) is quite poor. The finite difference solution with $a=1.1$ is consistently better than the OSRC method. However, the finite difference solution with $a=1.5$ is dramatically better. In Table 1 we present the $L_{2}$ errors of the normal derivative on the scatterer for an aspect ratio of 3.3 and various $k$. In Table 2 we instead consider an aspect ratio of 2 . For $k=1$ the OSRC of Reiner et al. works best among the standard OSRCs. For higher $k$ the various methods get closer though now Kriegsmann et al. followed by Reiner et al. are the best. For $k=1$ and an aspect ratio of 5 (not displayed) the OSRC of Reiner et al. is again best among the standard boundary conditions. Thus, when implemented as an OSRC the more sophisticated conditions that include curvature terms are less accurate than the simple models. However, the new method based on Mathieu functions is far superior especially for low and intermediate frequencies. In Table 3 we consider the Neumann problem (hard scatterer) with an aspect ratio of 2 . The error is now given for the solution on the scatterer. The conclusions are the same as for the Dirichlet boundary condition. In general, as an OSRC the best was the Mathieu modal expansion. The next best were the boundary conditions of Kriegsmann et al. and Reiner et al. The other boundary conditions that included terms that depend on the derivative of the curvature gave large errors unless $k$ was sufficiently large.

## 4. Conclusion

The new modal based boundary condition (20), imposed as an OSRC, is dramatically better than the other radiation conditions. The scheme of Reiner et al. was consistently reasonable. The OSRC of Kriegsmann et al. was more accurate than the generalizations that included curvature terms. The finite difference with a close in outer boundary $(a=1.1)$ is an improvement over the OSRC. With the outer boundary a little further out $(a=1.5)$ the FD is dramatically better than the OSRC.




Fig. 5. Dirichlet $\mathrm{bc}, k=4, \mathrm{AR}=2$.

Table 1
OSRC, Dirichlet bc, $L_{2}$ errors.

| AR $=3.3$ | $k=1$ | $k=2$ | $k=3$ |
| :--- | :--- | :--- | :--- |
| Grote | 1.625 | 1.418 | 1.217 |
| Reiner | 0.346 | 0.437 | 0.468 |
| Kriegsmann | 0.533 | 0.410 | 0.354 |
| Jones | 1.724 | 1.741 | 1.725 |
| Antoine | 1.657 | 1.615 | 1.576 |
| Kallivokas | 1.235 | 1.376 | 1.428 |
| Mathieu | 0.062 | 0.115 | 0.164 |

Table 2
OSRC, Dirichlet bc, $L_{2}$ errors.

| AR $=2$ | $k=1$ | $k=2$ | $k=3$ |
| :--- | :--- | :--- | :--- |
| Grote | 1.118 | 0.690 | 0.471 |
| Reiner | 0.228 | 0.282 | 0.298 |
| Kriegsmann | 0.244 | 0.193 | 0.190 |
| Jones | 0.517 | 0.635 | 0.320 |
| Antoine | 0.318 | 0.854 | 0.391 |
| Kallivokas | 0.311 | 0.450 | 0.281 |
| Mathieu | 0.058 | 0.089 | 0.113 |

Table 3
OSRC, Neumann bc, $L_{2}$ errors.

| AR $=2$ | $k=1$ | $k=2$ | $k=3$ |
| :--- | :--- | :--- | :--- |
| Grote | 1.645 | 1.241 | 1.070 |
| Reiner | 0.204 | 0.240 | 0.443 |
| Kriegsmann | 0.238 | 0.333 | 0.388 |
| Jones | 1.211 | 1.473 | 0.437 |
| Antoine | 1.717 | 1.799 | 0.525 |
| Kallivokas | 0.680 | 0.702 | 0.551 |
| Mathieu | 0.035 | 0.077 | 0.126 |

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