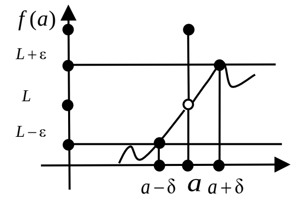
Handout 2



2 Limit of Function

Definition: The equation $\lim_{x\to a} f(x) = L$ means: "the limit of f(x) as *x* approaches *a* equals L".

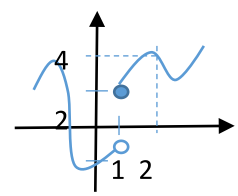
Note: The function f(x) does not have to be defined at x=a. It just has to get closer to L as x gets closer (but not equal) to a.

Definition: A more rigorous, (ε,δ)-definition, of limits. $\lim_{x\to a} f(x) = L$ if for each $\varepsilon > 0$ there is a $\delta > 0$ s.t. $x \in (a-\delta, a+\delta)$ implies $f(x) \in (L-\varepsilon, L+\varepsilon)$ (or, equivalently $|x-a| < \delta$ implies $|f(x)-L| < \varepsilon$).

Definition (*One Sided Limit*): The equation $\lim_{x\to a^+} f(x) = L$ means: "the limit of f(x) as x approaches a **from the right** equals L". Similarly $\lim_{x\to a^-} f(x) = L$ means: "the limit of f(x) as x approaches a **from the left** is L".

Theorem: A limit $\lim_{x\to a} f(x)$ is exists if and only if one-sided limits exists and equal, i.e. $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$.

2.1 Limit Laws



Definition: Let c be a constant and $\lim_{x\to a} f(x), \lim_{x\to a} g(x)$ exists and finite. Then

1) $\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

2) $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$

3) $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$

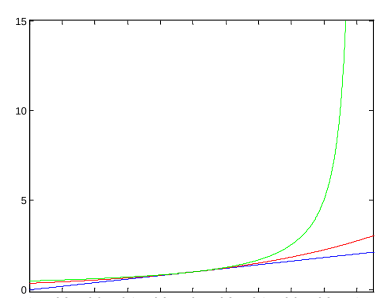
4) $\lim_{x\to a} f(x)/g(x) \underset{g(x)\neq 0}{=} \lim_{x\to a} f(x) / \lim_{x\to a} g(x)$

Direct Substitution Property: If f(x) "**have no problem at *a***" then $\lim_{x\to a} f(x) = f(a)$. For now we consider that roots, polynomial, rational and trigonometrical functions are such functions if defined at ***a***.

Definition: if $f(x) = g(x)$ in an open interval $x \in (a-b, a) \cup (a, a+b)$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$

Theorem: If $f(x) \le g(x)$ in an open interval $x \in (a-b, a+b)$, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$

The Squeeze Theorem (Other names: Sandwich Thrm, two policemen and a drunk theorem): If $f(x) \le g(x) \le h(x)$ in an open interval $x \in (a-b, a+b)$, and $\lim_{x\to a} f(x) = L = \lim_{x\to a} g(x)$, then $\lim_{x\to a} h(x) = L$.



3 Continuity

Definition: A function *f* is continuous at *x=a* if $\lim_{x\to a} f(x) = f(a)$

Note: The definition is implicitly require that x=a is in domain of definition of f(x) and that the limit is defined and equal f(a).

Definition: Interval is a set of real numbers with the property that any number that lies between two numbers in the set is also included in the set. Interval I can be either

- finite: $I = [b_1, b_2]$, $I = [b_1, b_2)$, $I = (b_1, b_2]$, $I = (b_1, b_2)$,
- half finite: $I = [b_1, \infty)$, $I = (-\infty, b_2]$, $I = (b_1, \infty)$, $I = (-\infty, b_2)$
- infinite interval: $I = (-\infty, \infty)$.

Definition: A function f is continuous on interval I if $\lim_{x\to a} f(x) = f(a)$ for every $a \in I$. In case of (half) finite interval we understand the definition as a one sided limit, i.e. $\lim_{x\to a^-} f(x) = f(a)$ or $\lim_{x\to a^+} f(x) = f(a)$.

Intuitively we understand continuity as no jump, hole, tear or break in the graph of the function.

Theorem: Let f(x), g(x) be continuous at x=a, and let c be a constant. The following functions are continuous: $f(x) \pm g(x)$, $cf(x)$, $f(x)g(x)$ and $\frac{f(x)}{g(x)}$ if $g(a) \neq 0$

Corollary:

- Any polynomial is continuous function on $\mathbb{R} \in (-\infty, \infty)$.
- Any rational function is continuous on its domain of definition.

Theorem: A following functions are continuous on their domain of definition: $\sqrt{x}, e^x, \ln x$ and trig functions.

Theorem: If f(x) is continuous at x=b and $\lim_{x\to a} g(x) = b$ then $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$.

Corollary: If $g(x)$ is continuous at $x = a$ and $f(x)$ is continuous at $x = g(a)$, then the composition function $(f \circ g)(x) = f(g(x))$ is continues at x=a;

Classification of discontinuities:

1) **Removable discontinuity**: One sided limits at $x = a$ exists, finite and equal, but not equal to $f(a)$ (this is a function with hole, which can be removed, aka "filled").
2) **Step/Jump discontinuity**: One sided limits at $x = a$ are exists and finite **but not equal**.
3) **Essential discontinuity**: At least one of one-sided limits not exists or infinite.

Theorem (The intermediate value theorem): Let $f(x)$ be continuous function on closed interval $I = [a, b]$ and let $N \in [f(a), f(b)]$ (or $N \in [f(b), f(a)]$), there is exists $c \in [a, b]$ such that $f(c) = N$.