Course: MA141, Calculus I	Instructor Michael Medvinsky

Handout 4

4 Derivatives

The tangent line to the curve y=f(x) at the point $(a, f(a))$ is the line $l(x) = mx + b$ through this point with slope $m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$ provided that this limit exists. Alternatively, one can express the slope as $m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

Instantaneous vs. Average Velocity: If you drive a car from SLC to Las Vegas, your average velocity is given by $\frac{\text{displacement}}{\text{time}} = \frac{f(t+h) - f(t)}{h}$ (note that velocity mean speed and direction, i.e. it can be either negative or positive). However when you ride the car your speedometer shows you different values of the speed\velocity during the trip, because it represents an instantaneous velocity, which can be expressed by $v(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$.

Rate of change: Denote $\Delta x = x_1 - x_2$ the change in x and $\Delta y = y_1 - y_2 = f(x_1) - f(x_2)$ the change in y. The rate of change is $\frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$ and instantaneous rate of change is $\lim_{x_1\to x_2} \frac{f(x_1) - f(x_2)}{x_1 - x_2}$

The Derivative of f(x) at x=a is denoted as $f'(a) = \left.\frac{df(x)}{dx}\right|_{x=a}$. $f'(a)$ is an instantaneous rate of change if y=f(x) with respect to x when x=a, i.e. $f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$ provided the limit exists. It can be of course alternatively defined as $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(a) - f(a-h)}{h}$ and even $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h}$. Similarly, for any $x \in \mathbb{R}$ we get a function $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$.

Definition: A function f is **differentiable at x=a** if f'(a) exists. It is **differentiable on an open interval** (a,b) (can also be half- inf or inf) if it is differentiable at every $x \in (a,b)$.

Theorem: If f is differentiable at a then f is continuous at a.

Note: every differentiable is continuous, but not every continuous function is differentiable

Definition: A **Necessary Condition** for some statement S is a condition that must be satisfied in order for S to obtain.

Definition: A **Sufficient Condition** for some statement S is a condition that, if satisfied, guarantees that S obtains.

Corollary: Continuity is a necessary, but not sufficient condition for differentiability of a function.

Corollary: Differentiability is sufficient, but not necessary condition for continuity of a function.

4.1 Derivative rules

Let's f(x) and g(x) be differentiable functions. Let also $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$.

Name	rule
Constant rule	$\frac{d}{dx}\alpha = 0$
Power Rule	$\frac{d}{dx}x = 1$
	$\frac{d}{dx}x^n = nx^{n-1}$
	$\frac{d}{dx}x^\alpha = \alpha x^{\alpha-1}$
Rule of Sum	$\frac{d}{dx}\{f(x)+g(x)\} = f'(x)+g'(x)$
Product Rule	$\frac{d}{dx}\{\alpha f(x)\} = \alpha f'(x)$
	$\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x)+f(x)g'(x)$
Quotient Rule	$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain Rule	$\frac{d}{dx}\underbrace{f}_{\substack{\text{outer}\\ \text{function}}}\underbrace{(g(x))}_{\substack{\text{inner}\\ \text{function}}} = \underbrace{f'(g(x))}_{\substack{\text{derivative of outer}\\ \text{function evaluated}\\ \text{at inner function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative of}\\ \text{inner function}}}$
Chain+Power rules	$\frac{d}{dx}\left\{(g(x))^n\right\} = n(g(x))^{n-1}g'(x)$
Exponential rule	$\frac{d}{dx}e^x = e^x$
	$\frac{d}{dx}\alpha^x = \alpha^x \ln\alpha$

4.2 Higher Derivatives

Since the derivative of a function f(x) is a function f'(x) we can derive it again (given it is differentiable). We denote following derivation as

1) $f'(x)=\frac{d}{dx}f(x)=\frac{df(x)}{dx}$

2) $f''(x)=\left(\frac{d}{dx}\right)^2 f(x)=\frac{d^2}{dx^2}f(x)=\frac{d}{dx}f'(x)$

3) $f^{(n)}(x)=\left(\frac{d}{dx}\right)^n f(x)=\frac{d^n}{dx^n}f(x)=\frac{d}{dx}f^{(n-1)}(x)$

Example: $(x^n)^{(n)}=n(x^{n-1})^{(n-1)}\ldots=n\cdots(n-k+1)(x^{n-k})^{(n-k)}=\frac{n!}{(n-k)!}(x^{n-k})^{(n-k)}\ldots=\frac{n!}{\not{0}!}\not{x^0}$

5 What does f', f''say about f? (2.8)

Derivatives are very important thing in calculus. One can learn a lot about function f(x) using information about its derivatives. For example the sign of f' provides us information about direction of the function:

If $f'(x)>0$ on some interval, then f is **increasing** there.

If $f'(x)<0$ on some interval, then is **decreasing** there.

Furthermore, the second derivative can provide even more information:

If $f''(x)>0$ on some interval, then f is **concave upward** (happy smile) there.

If $f''(x)<0$ on some interval, then f is **concave downward** (convex, sad smile) there.

6 Miscellaneous Definitions:

Def (Even function): f(x)=f(-x), i.e. it is symmetric with respect to the y-axis, thus the graph remains unchanged after reflection about the y-axis.

Def (Odd function): f(x) = -f(-x) i.e. it has rotational symmetry with respect to the origin, thus the graph remains unchanged after rotation of 180 degrees about the origin.

Antiderivatives: Some time we know f(x) and we need to find another function F(x) such that F'(x)=f(x). If such F(x) exists we call it antiderivative. Since f(x) is derivative of F(x) we have lot information about F(x) to work with. We'll learn it in future.

Binomial Theorem:

$$(x+y)^n=x^n+nx^{n-1}y+\frac{n(n-1)}{2}x^{n-2}y^2+\ldots+\frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k}x^{n-k}y^k+\ldots+nxy^{n-1}+y^n$$