Handout 7

13 Optimization Problems

- **Definition**: Optimization problem is a problem of finding the best, a.k.a. optimal, solution or value. Optimization problem is often formulated as a problem of finding maximum or minimum of a function, e.g. maximizing the benefits or minimizing costs.

13.1 Newton's Method

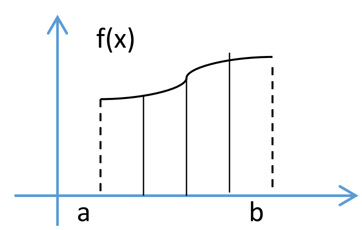
In order to solve optimization problem one needs to find critical points, and therefore needs to know to find roots\zeros of functions. In a real life optimization problem there is often no analytical way to find the root of the function of the interest, hence one approximates the root. This can be done, for example, using Newton-Raphson method, often shortened as a **Newton method**.

Newton method: To find solution to $f(x)=0$, make an initial guess x_0 then improve the guess using iteration $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$ until $\left|x_{n+1}-x_n\right|=\left|\frac{f(x_n)}{f'(x_n)}\right|<\varepsilon$, where ε which bounds the error of the approximation has to be small enough for the problem of the interest.

Note: The initial guess, x_0, has to be "close enough" to the root, otherwise the method may fail. *Take Numerical Analysis course for more information.*

14 Integrals

Consider an interval $I=[a,b]$ divided into n subintervals $I_j=[x_{j-1},x_j]$, where $x_j=a+j\Delta x$ and $\Delta x=x_j-x_{j-1}=\frac{b-a}{n}$, so that $a=x_0<x_1<...<x_n=b$. In addition, denote 2 points in the subinterval I_j: (1) $x_j^*\in I_j$ an arbitrary/**sample** point, and (2) a midpoint $\bar{x}_j=\frac{x_{j-1}+x_j}{2}\in I_j$.



Definition: A function *F*(*x*) is called an **antiderivative** of the function *f*(*x*) on an interval *I* if *F'*(*x*)=*f*(*x*) for all *x* on *I*.

Theorem: If *F*(*x*) is an antiderivative of *f*(*x*) on an interval *I,* then for any constant *C, F*(*x*)+*C* is also an antiderivative of *f*(*x*).

Theorem: The area of the region that lies under *f*(*x*), $x \in I$ is given by

$$A = \lim_{n\to\infty}\left\{\left(f\left(x_1\right)\Delta x\right)+\left(f\left(x_2\right)\Delta x\right)+...\left(f\left(x_n\right)\Delta x\right)\right\} = \lim_{n\to\infty}\sum_{j=1}^{n} f\left(x_j\right)\Delta x .$$

Definition: Denote **Riemann sum** by $S = \lim_{n\to\infty}\sum_{j=1}^{n} f\left(x_j^*\right)\Delta x$.

Definition: The **definite integral of *f* from *a* to *b*** is $\int_a^b f\left(x\right)dx = \lim_{n\to\infty}\sum_{j=1}^{n} f\left(x_j^*\right)\Delta x$ provided that the limit exists. If the limit exists we say that f is **integrable** on [a,b].

Notes:

- $\int$ is an integral sign, *f*(*x*) is an **integrand** and a, b are lower and upper limits of the integral respectively. Evaluating\calculating the integral is called integration.
- The integral is not depends on x, i.e. $\int_a^b f\left(x\right)dx = \int_a^b f\left(r\right)dr = \int_a^b f\left(t\right)dt$
- If *f*(*x*)>0 in [*a,b*], then an integral represent the area that lies under *f*(*x*). For *f*(*x*)<0 in [*a,b*], the integral represent –A of -*f*(*x*). If *f* changes signs, then it represent the difference between areas of negative and positive regions.
- The Reimann sum can be equivalently defined using intervals of unequal width.

Theorem: If *f* is continuous on [a,b], or if it has finite number of jump discontinuities, then *f* is integrable.

Theorem: If *f* is integrable on [a,b] then $\int_a^b f\left(x\right)dx = \lim_{n\to\infty}\sum_{j=1}^{n} f\left(x_j\right)\Delta x$.

Integral Approximation (Rectangular) Methods: Let $N \in \mathbb{N}$

- **Midpoint Rule** $f\left(x\right) \approx \sum_{j=1}^{N} f\left(\overline{x}_j\right)\Delta x = \frac{b-a}{N}\sum_{j=1}^{N} f\left(\frac{x_{j-1}+x_j}{2}\right)$
- **Endpoint rule** $f\left(x\right) \approx \frac{b-a}{N}\sum_{j=1}^{N} f\left(x_{i-1}\right) \approx \frac{b-a}{N}\sum_{j=1}^{N} f\left(x_i\right)$