Course: MA141, Calculus I	Instructor Michael Medvinsky

Handout 9

15 Application of Integration

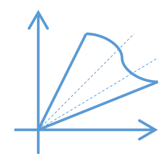
Theorem: Let $f(x) > g(x)$ be continuous functions for all $x \in [a,b]$. The area of region bounded by curves y=f(x), y=g(x) and the lines x=a and x=b is given by

$$A = \int_a^b f(x) - g(x)\,dx$$

Note: If f(x) and g(x) are velocities of moving objects and f(a)=g(a) the area is interpreted as distance between the objects at time x=b.

Definition: Polar coordinates are given by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\dfrac{y}{x} \end{cases}$$



Definition: A straight line (a ray) that passes through origin and making angle θ_0 with polar axis is given by formula $\theta = \theta_0$.

15.1 Area

- The area enclosed between θ_1 and θ_2 and the constant radius r in polar coordinates is

$$A = \frac{1}{2} r^2 (\theta_2 - \theta_1).$$

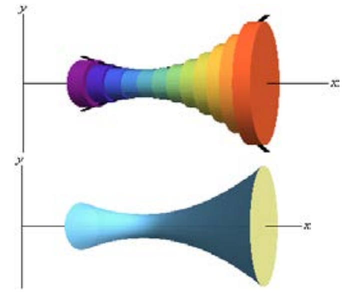
- Area enclosed between θ_1 and θ_2 and a function $r = f(\theta)$ in polar coordinates one splits the interval $[\theta_1, \theta_2]$ into n pieces $[\theta_{i-1}, \theta_i]$ where $\theta_i = \theta_1 + i\dfrac{\theta_2 - \theta_1}{n}$, such that $\Delta\theta = \theta_i - \theta_{i-1}$ is small enough so that $r_i^* = f(\theta_i^*)$ can be considered constant for an arbitrary $\theta_i^* \in [\theta_{i-1}, \theta_i]$, then

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} r_i^* (\theta_i - \theta_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} f^2(\theta_i^*)\Delta\theta = \int_{\theta_1}^{\theta_2} \frac{1}{2} f^2(\theta)\, d\theta$$

15.2 Volume

Volume of a simple body like circular\elliptical cylinder or rectangular parallelepiped (a box) is

$$Volume = Area\ of\ Basis \times height = Ah$$



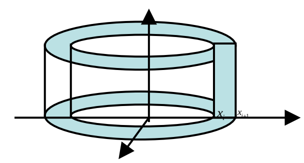
Solid of Revolution: Volume of arbitrary body B one cut/slice it into small cylindrical pieces and approximate it by sum of the volumes of the small constant cylinders.

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} A\left(x_i^*\right)\Delta x = \int_a^b A(x)\,dx \quad \text{or} \quad V = \int_c^d A(y)\,dy\,,$$

where $A\left(x_i^*\right), x_i^* \in \left[x_{i-1}, x_i\right]$ is the area of a cross section of the slice $\left[x_{i-1}, x_i\right]$.

Area of Disc: $A = \pi r^2$, where $r = f(x)$ or $r = f(y)$

Area of washe*r\annular ring*: $A = \pi\left(r_{outer}^2 - r_{inner}^2\right)$



Method of Cylindrical shells: Instead of slicing into discs/washers one divides\peels the body into cylindrical shells. In case of rotation of *f*(*x*) around *y*, *f*(*x*) *become a heght* and the volume of every shell is given by $V_i = 2\pi x_i^* f\left(x_i^*\right)\Delta x$ so that

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi x_i^* f\left(x_i^*\right)\Delta x = \int_a^b 2\pi x f(x)\,dx$$

15.3 Arc Length

The length of curve given by

- $y = f(x),\ x \in (a,b)$ is $L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx$
- $x = g(y),\ y \in (c,d)$ is $L = \int_c^d \sqrt{1 + \left(g'(y)\right)^2}\,dy$
- $(x,y) = \left(f(t), g(t)\right), t \in (a,b)$ is $L = \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\,dt$
- $r = f(\theta)$ (polar) is $L = \int_a^b \sqrt{r^2 + \left(f'(\theta)\right)^2}\,d\theta$

15.4 Average Value of Function

Theorem: Let *f*(*x*) be continues on [a,b]. There is exists a $c \in [a,b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$.