

10.9 Rates of Change in Natural/Social Sciences

Recall: **Rate of change**: denote change in x as $\Delta x = x_1 - x_2$ and change in y as

$\Delta y = y_1 - y_2 = f(x_1) - f(x_2)$, then the rate of change is given

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$
 and instantaneous rate of change is $\lim_{x_1 \rightarrow x_2} \frac{f(x_1) - f(x_2)}{x_1 - x_2}$.

Physics

1. **Motion of particle** was in lab and practice exam
2. If a rod or piece of wire is homogeneous, then its **linear density** is uniform and is defined as the mass per unit length $\rho = m/l$ and measured in kilograms per meter. In this problem we consider inhomogeneous rod which mass measured from its left end to a point x is $m = f(x)$.

Therefore, the mass of the part of the rod that lies between x_1 and x_2 is given by

$\Delta m = f(x_2) - f(x_1)$. The average density $= \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ and the linear

density at x_1 is $\rho = \lim_{\Delta x \rightarrow 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$. Thus the linear density of the rod is derivative of the mass with respect to length.

3. If ΔQ is the net charge that passes through a plane surface during time period Δt then an average (electric) current $= \frac{\Delta Q}{\Delta t}$ and the current is $I = \lim_{\Delta x \rightarrow 0} \frac{\Delta Q}{\Delta t}$, which measured in amperes (=coulombs per second).
4. Additional uses in physics includes power, heat flow and not only

Chemistry

Ex 1. If a materials (reactants) A and B, produce by a chemical reaction, a substance C (a product). The concentration of reactant A is given in moles per liter (1 mole = 6.022×10^{23} molecules) and denoted as $[A]$. The concentration varies during a reaction (i.e. it's a function of time). So the average **rate of reaction** is $\frac{\Delta [A]}{\Delta t} = \frac{[A](t_2) - [A](t_1)}{t_2 - t_1}$ while the instantaneous on is given $\lim_{\Delta t \rightarrow 0} \frac{\Delta [A]}{\Delta t} = \frac{d[A]}{dt}$. This

is also satisfying for example if $A + B \rightarrow C$ then $-\frac{d[A]}{dt} = -\frac{d[B]}{dt} = \frac{d[C]}{dt}$, where the minus represent decreasing concentration whenever plus increasing.

Furthermore for a reaction $aA+Bb \rightarrow cC+dD$ one get

$$-\frac{1}{a} \frac{d[A]}{dt} = -\frac{1}{b} \frac{d[B]}{dt} = \frac{1}{c} \frac{d[C]}{dt} = \frac{1}{d} \frac{d[D]}{dt}$$

Ex 2. Isothermal **compressibility** is given by $\beta = -\frac{1}{V} \frac{dV}{dP}$. In constant temperature if the pressure P is increased then the volume V is decreased (i.e. $\frac{dV}{dP} < 0$). Thus β measures how fast per unit volume, the volume of substance decreases as the pressure increases at constant temperature.

Biology

Ex 3. If the population size of an animal or plant is changes between the times one define the average rate of growth as $\frac{\Delta n}{\Delta t}$ where Δn is difference in population and t is the time. The instantaneous **rate of growth** is given by $\frac{dn}{dt}$. Note: the population growth isn't continues function, so it is not differentiable. However for a large population we change the discontinuous function with a smooth approximation curve, which is differentiable.

Ex 4. Consider that a vein or artery is approximated by cylindrical tube of radius R and length l . The velocity v of blood is greatest along the central axis and decreases as it get close to the walls, this is because of friction. The **Law of Laminar (blood) Flow**, which was discovered by physician Jean-Louis-Marie Poiseuille in 1840 is given by $v = \frac{P}{4\eta l} (R^2 - r^2)$, r is the distance from the axis, P is the pressure difference between the ends of the tube and η is the viscosity of the blood. The rate of change of velocity from center to walls (when considering everything but r is constant) is given by $v' = \frac{P}{4\eta l} (-2r) = -\frac{P}{2\eta l} r$.

Economics

Ex 5. The cost function $C(x)$ is the price that a company incurs in producing x units of some commodity. The **marginal cost** is an instantaneous rate of change of cost with respect to the number of items produced, i.e. $\frac{dC}{dx}$.

Note that usually x is an integer; therefore we often change $C(x)$ by an approximation. The approximation is often polynomial.

Other sciences

Psychology: take a learning performance as a function of training time – the rate of skills improvement.

Sociology: Propagation of rumor during time - a rate of spread.

10.10 Linear Approximation

A linear approximation exploit the fact that the function and the tangent line at $x=a$ are very close near a .

The tangent line is given by $l(x) = f'(a)x + b$ so that

$$\begin{aligned} l(a) &= f'(a)a + b = f(a) \Rightarrow b = f(a) - f'(a)a \\ \Rightarrow l(x) &= f'(a)x + f(a) - f'(a)a = f'(a)(x-a) + f(a) \end{aligned}$$

Thus, the linear approximation is given by $f(x) \approx f(a) + f'(a)(x-a)$ or equivalently $f(a+h) \approx f(a) + hf'(a)$ (here $x = a+h$ and $h = x-a$). One can guess that the smaller the h the better the linear approximation approximates $f(x) = f(a+h)$.

The estimated error of linear approximation is bounded by $E(h) = \frac{f''(c)}{2}h^2$ where $c \in (a, a+h)$ (c isn't known, so we have to maximize it to estimate the error).

The absolute error is given by the absolute value of the difference between exact value and approximation $E = |f(a+h) - (f(a) + hf'(a))| = |\Delta f - hf'(a)|$.

Ex 6. Using calculator one can see that $\sqrt{2} \approx 1.41421$. We now see how far the linear approximation from it: $\sqrt{2} \approx \sqrt{1} + 1 \cdot \frac{1}{2\sqrt{1}} = 1 + \frac{1}{2} = 1.5$. The formula for error give us

$$\begin{aligned} \frac{d^2}{dx^2}\sqrt{x} &= \frac{d}{dx} \frac{1}{2\sqrt{x}} = -\frac{2/2\sqrt{x}}{4x} = -\frac{1}{4x^{3/2}} \\ \Rightarrow |E| &= \left| \frac{f''(a)}{2}h^2 \right| \leq \left| \frac{1}{2} \left(-\frac{1}{4 \cdot 1^{3/2}} \right) 1^2 \right| = -\frac{1}{8} \approx -0.125 \end{aligned}$$

Whereas the absolute error is $|1.5 - 1.41| = 0.09 < 0.125$.

Ex 7. We now approximate $\sqrt{4+h}$

$$\begin{aligned}\sqrt{4+h} &\approx \sqrt{4} + \frac{1}{2\sqrt{4}}h = 2 + \frac{1}{4}h = \frac{8+h}{4} \\ E &= \frac{1}{2} \left(-\frac{1}{4 \cdot 4^{3/2}} \right) h^2 = -\frac{1}{64}h^2\end{aligned}$$

Let evaluate several values of h $h=0.1$ gives 2.025 and the error =0.00015625

h	exact	LA	$ \sim\text{err} $	$ \text{err} $
0.1	2.0248	2.025	1.5625e-4	1.5432e-4
0.2	2.0493	2.050	6.2500e-4	6.0984e-4
0.4	2.0976	2.100	25.000e-4	23.8230e-4

Ex 8. $\sin(x+h) \approx \sin x + h \cos x$, when $x=0$ and h is small enough we get

$$\sin h \approx h \Rightarrow \frac{\sin h}{h} \approx 1.$$

10.10.1 Differentials

Let $y = f(x)$ then

$$\begin{aligned}f(x+\Delta x) &\approx f(x) + \Delta x f'(x) \\ \Delta y &= f(x+\Delta x) - f(x) \approx \Delta x f'(x) \\ \frac{\Delta y}{\Delta x} &\approx f'(x)\end{aligned}$$

Thus, if we take Δx sufficiently close to 0 we get $dy = f'(x)dx$ and then one can write:

$$f(x+dx) \approx f(x) + dx \frac{dy}{dx} = f(x) + dy$$

Ex 1. Consider that p is a measured input of the equation $x^2 - 2px + 1 = 0$. The solution is given as $x_{\pm}(p) = p \pm \sqrt{p^2 - 1}$. Since the input is measured we denote the error of measure by dp . Then the error in $x_{\pm}(p)$ will be

$$dx_{\pm}(p) = \left(1 \pm \frac{p}{\sqrt{p^2 - 1}} \right) dp. \text{ The 2 solutions } x_{\pm}(p) \text{ are getting close as } p \rightarrow 1, \text{ and the}$$

approximation blows up when $p=1$. Thus, the solution is very sensitive to the error dp in p when p is very close to $p=1$. Which means small error in p lead to big error in $x_{\pm}(p)$ near $p=1$.