## 11.5 Optimization Problems(4.6)

Ex 1. A garden has 200 pounds of watermelons growing in it. Every day, the total amount of watermelon increases by 5 pounds. At the same time, the price per pound of watermelon goes down by 1¢. If the current price is 90¢ per pound, how much longer should the watermelons grow in order to fetch the highest price possible?

**Solution**: After $x$ days, there will be $200 + 5x$ pounds of watermelon, which will valued at $90 – x$ cents per pound. Thus, the price after $x$ days will be Price($x$) = $(200 + 5x)(90 – x)$ cents. The derivative is Price'($x$) = $250 – 10x$, which is zero when $x = 25$. Because Price'($x$) is clearly positive when $x$ is less than 25 and negative afterward, this is maximal by the first derivative test. Thus, the watermelons will fetch the highest price in 25 days.

Ex 2. When 30 orange trees are planted on an acre, each will produce 500 oranges a year. For every additional orange tree planted, each tree will produce 10 fewer oranges. How many trees should be planted to maximize the yield?

**Solution**: If $x$ is the number of trees beyond 30 that are planted on the acre, then the number of oranges produced will be: Oranges($x$) = (number of trees) (yield per tree) = $(30 + x)(500 – 10x) = 15{,}000 + 200x – 10x^2$. The derivative Oranges'($x$) = $200 – 20x$ is zero when $x = 10$. Using the second derivative test, Oranges"($x$) = –20 is negative, so this is maximal. Thus, $x = 10$ more than 30 trees should be planted, for a total of 40 trees per acre.

## 11.6 Newton's Method (4.7)

Consider an optimization problem represented by high order polynomial. To find a minimum or maximum of this problem we have to find zeros of the first derivatives which is also high order polynomial (one degree less than original function). Unfortunately, it has been proved in 19 century that there is no analytical solution to finding zeros of polynomial above degree 5. When the function isn't polynomial finding zeros could be even more complicated. Thus we looking for approximation of an equation $f(x)=0$. Note, if we want to solve $f(x)=d$ we can simply solve $f(x)-d=0$.

In this course we will learn one specific iterative method (a sequence of improving approximate solutions), a method of Newton-Raphson, often shortened as a **Newton method**: Let $r$ be solution of $f(x)=0$, i.e. $f(r)=0$. Consider $x_n$ approximates $r$, i.e. $x_n \approx r$, thus $f(x_n)\approx 0$. Assume now a linear approximation $0 \approx f(x_{n+1}) \approx f(r) + f'(r)(x_{n+1}-r)$ and solve it for $x_{n+1} \approx r - \frac{f(r)}{f'(r)}$. Since $r$ is unknown we approximate it with $x_n \approx r$ and define the iteration $x_{n+1} \approx x_n - \frac{f(x_n)}{f'(x_n)}$.

**Termination**: We continue the iteration until absolute error $|x_{k+1}-x_k|$ gets small enough or since $x_{k+1}-x_k = -\frac{f(x_k)}{f'(x_k)}$ we can also use $\left|\frac{f(x_k)}{f'(x_k)}\right| \le \varepsilon$ ( $\varepsilon > 0$ denotes small number).

**Note**: The initial guess, $x_0$, has to be "close enough" to $r$, otherwise the method may fail. Take Numerical Analysis course for more information.

Ex 1. Let $f(x) = x^2 - \frac{1}{3}, x \in [0,1]$

a. Write Newton iteration
b. For initial guess $x_0 = 1$ do 3 iterations, write the error each step

i. $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - \frac{1}{3}}{2x_n} = \frac{2x_n^2 - x_n^2 + \frac{1}{3}}{2x_n} = \frac{x_n^2 + \frac{1}{3}}{2x_n} = \frac{x_n}{2} + \frac{1}{6x_n}$$

ii.
$$x_0 = 1 \qquad \Rightarrow e_0 \approx 0.4226$$
$$x_1 = \frac{1}{2} + \frac{1}{6} = \frac{3+1}{6} = \frac{4}{6} = \frac{2}{3} \approx 0.6667 \qquad \Rightarrow e_1 \approx 0.0893$$
$$x_2 = \frac{1}{2}\frac{2}{3} + \frac{1}{6}\frac{3}{2} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \approx 0.5833 \qquad \Rightarrow e_2 \approx 0.00598$$
$$x_3 = \frac{7}{24} + \frac{2}{7} = \frac{97}{168} = 0.577381 \qquad \Rightarrow e_3 \approx 3.06832 \cdot 10^{-5}$$

Ex 2. Let $x + \ln x = 0$ show first 3 steps ($x_1, x_2, x_3$) of newton iteration for initial guess a) x=4 and b) x=1:

a) $x_1 = x_0 - \frac{x_0 + \ln x_0}{1 + 1/x_0} = 4 - \frac{4 + \ln 4}{1 + 1/4} = -0.3090$ so we can't continue b)

$$x_1 = x_0 - \frac{x_0 + \ln x_0}{1 + 1/x_0} = 1 - \frac{1 + \ln 1}{1 + 1/1} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$x_2 = \frac{1}{2} - \frac{\frac{1}{2} + \ln\frac{1}{2}}{1 + 2} = 0.5644 \qquad x_3 = 0.5671$$

## 11.7 Antiderivatives (4.8)

**Def**: A function $F(x)$ is called an antiderivative of $f(x)$ on an interval $I$ if $F'(x)=f(x)$ for all $x$ on $I$.

**Thm**: If $F$ is antiderivative of $f$ on an interval $I$, then for any constant $C$, $F(x)+C$ is a (most general) antiderivative.

Ex 3. Antiderivative of $3x^2$ is $x^3 + C$
Ex 4. Antiderivative of $\cos x$ is $\sin x + C$
Ex 5. Antiderivative of $\sin x$ is $-\cos x + C$
Ex 6. Antiderivative of $e^x$ is $e^x + C$
Ex 7. Antiderivative of $1/x$ is $\ln |x| + C$
Ex 8. Antiderivative of $af(x)$ is $aF(x) + C$
Ex 9. Antiderivative of $f(x) + g(x)$ is $F(x) + G(x) + C$
Ex 10. Find $f$ if $f''(x) = x^2 + 2x - 1, f(0) = -\frac{1}{12}, f(1) = \frac{1}{3}$

**Solution**: We first find $f'(x) = \frac{1}{3}x^3 + x^2 - x + C_1$ and next we find $f(x) = \frac{1}{3} \cdot \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + C_1 x + C_2$. Now we use the additional information to find the constants: $f(0) = C_2 = -\frac{1}{12}$ and

$f(1) = \frac{1}{12} + \frac{1}{3} - \frac{1}{2} + C_1 - \frac{1}{12} = \frac{1}{3}$, so $C_1 = \frac{1}{2}$