# 12 Integrals
12.1 Areas and Distances (5.1)

When we measure the area of simple shape like rectangular ($A = ab$) or triangle ($A = \frac{1}{2}bh$) it is quite simple, but when the shape become more complicate (e.g. octagon) one divide it into (non overlapping) more simple shapes (e.g. triangles and rectangles) which areas are known and sum up the areas of these shapes to get the area of the complicate one.

Consider we measure the area of between the function f(x) and the x-axis on an interval [a,b]. This situation is even more complicate. First we can divide, the area of interest into as many shapes as we wish, but the resultant shape may still have similar problem. One overcomes the difficulty by approximating the shape by either smallest bounding or bounded rectangular with one of the upper corners touches the function, i.e. $f(x_0)$ or $f(x_1)$.
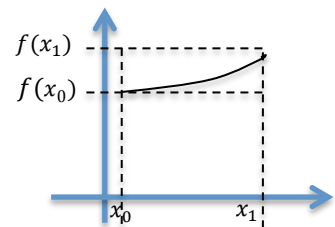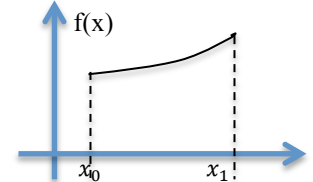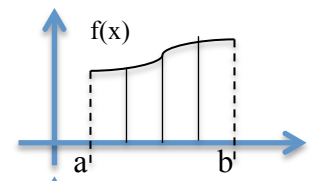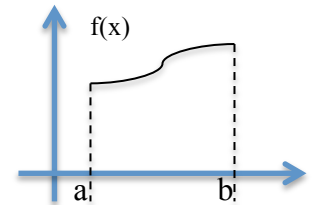
Ex 1. Estimate the area of the region that lies under $f(x) = \sqrt{1-x^2}$ on $[0,1]$ by dividing the interval into 2, 3 and 4 pieces. Compare to $\frac{\pi r^2}{2}$.

Solution:

1. First we divide the interval into [0,0.5] and [0.5,1], then the area is

   approximated by $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{4-1}{4}} = \frac{1}{2} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{2+\sqrt{3}}{4} \approx 0.933$

2. First we divide the interval into $\left[0,\frac{1}{3}\right], \left[\frac{1}{3},\frac{2}{3}\right]$ and $\left[\frac{2}{3},1\right]$, then the area is

   approximated by $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \sqrt{1-\left(\frac{1}{3}\right)^2} + \frac{1}{3} \cdot \sqrt{1-\left(\frac{2}{3}\right)^2} = \frac{1}{3} + \frac{1}{3} \cdot \frac{\sqrt{8}}{3} + \frac{1}{3} \cdot \frac{\sqrt{5}}{3} \approx 0.896$

3. First we divide the interval into $\left[0,\frac{1}{4}\right],\left[\frac{1}{4},\frac{2}{4}\right],\left[\frac{2}{4},\frac{3}{4}\right]$ and $\left[\frac{3}{4},1\right]$, then the area is approximated by

$$\frac{1}{4}\cdot 1+\frac{1}{4}\cdot\sqrt{1-\left(\frac{1}{4}\right)^2}+\frac{1}{4}\cdot\sqrt{1-\left(\frac{2}{4}\right)^2}+\frac{1}{4}\cdot\sqrt{1-\left(\frac{3}{4}\right)^2}=\frac{1}{4}+\frac{1}{4}\cdot\frac{\sqrt{15}}{4}+\frac{1}{4}\cdot\frac{\sqrt{12}}{4}+\frac{1}{4}\cdot\frac{\sqrt{7}}{4}\approx 0.874$$

4. An exact area is given by $\dfrac{\pi r^2}{4}=\dfrac{\pi}{4}=0.785$

To find an exact area using the method above one actually need to solve the

following limit $A=\lim\limits_{n\to\infty}\left(\dfrac{1}{n}\cdot 1+\dfrac{1}{n}\cdot\sqrt{1-\left(\dfrac{1}{n}\right)^2}+\dfrac{1}{n}\cdot\sqrt{1-\left(\dfrac{2}{n}\right)^2}+\cdots+\dfrac{1}{n}\cdot\sqrt{1-\left(\dfrac{n-1}{n}\right)^2}\right).$

Thm: The area of the region that lies under $f(x)$ is given by

$$A=\lim_{n\to\infty}\left(f\left(x_1\right)\Delta x+f\left(x_2\right)\Delta x+\cdots+f\left(x_n\right)\Delta x\right)=\lim_{n\to\infty}\sum_{j=1}^{n}f\left(x_j\right)\Delta x$$

More generally, let x_j^* be an arbitrary point such that $x_j^*\in\left(x_{j-1},x_j\right)$, then

$A=\lim\limits_{n\to\infty}\sum\limits_{j=1}^{n}f\left(x_j^*\right)\Delta x$ this called **Reimann sum**.