12.2 The Definite Integrals (5.2)

Def: Let f(x) be defined on interval [a,b]. Divide [a,b] into n subintervals of equal width $\Delta x = \frac{b-a}{n}$, so $x_0 = a, x_1 = a + \Delta x, x_j = a + j\Delta x, x_n = b$. Let x_j^* be an arbitrary (sample) points such that $x_j^* \in (x_{j-1}, x_j)$. Then the **definite integral of *f* from *a* to *b*** is

$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{j=1}^n f(x_j^*)\Delta x$ provided that the limit exists. If it douse exist, we say that *f* is **integrable** on [*a,b*].

Notes:

- $\int$ is an integral sign, *f*(*x*) is an **integrand** and a, b are lower and upper limits of the integral respectively. Evaluating\calculating the integral is called integration.
- The integral is not dependend on x, i.e. $\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$
- If *f*(*x*)>0 in [*a,b*], then an integral represent the area that lies under *f*(*x*). For *f*(*x*)<0 in [*a,b*], the integral represent –A of -*f*(*x*). If *f* changes signs, then it represent the difference between areas of negative and positive regions.
- The Reimann sum can be equivalently defined using intervals of unequal width.

Ex 2. Evaluate $\int_0^3 x-2\,dx$.

Solution: Sketch the graph of x-2 and figure out that the function is crossing x-axis, therefore the integral is the difference of areas (of 2 triangles). Thus $\int_0^3 x-2\,dx = \frac{1}{2}(1\cdot 1) - \frac{1}{2}(2\cdot 2) = \frac{1}{2} - 2 = -1.5$.

Thm: If *f* is continuous on [a,b], or if it has finite number of jump discontinuities, then *f* is integrable.

Thm: If *f* is integrable on [a,b] then $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{j=1}^n f(x_j)\Delta x$ where $\Delta x = \frac{b-a}{n}$ and $x_0 = a, x_1 = a + \Delta x, x_j = a + j\Delta x, x_n = b$.

12.2.1 Evaluating Integrals

Ex 3. $\int_a^b 1\,dx = \int_a^b dx = \lim_{n\to\infty}\sum_{j=1}^{n} 1\cdot\Delta x = \lim_{n\to\infty}\sum_{j=1}^{n}\Delta x = \lim_{n\to\infty}\Delta x\sum_{j=1}^{n} 1 = \lim_{n\to\infty} n\Delta x = \lim_{n\to\infty} n\frac{b-a}{n} = (b-a)$

Ex 4. $\int_a^b x\,dx = \lim_{n\to\infty}\sum_{j=1}^{n} x_j\Delta x = \lim_{n\to\infty}\sum_{j=1}^{n}(a+j\Delta x)\Delta x = \lim_{n\to\infty}\sum_{j=1}^{n}\left(a\Delta x + j(\Delta x)^2\right) =$

$$= \lim_{n\to\infty}\left(a\Delta x\sum_{j=1}^{n} 1 + (\Delta x)^2\sum_{j=1}^{n} j\right) = \lim_{n\to\infty}\left(an\Delta x + (\Delta x)^2\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left(an\frac{b-a}{n} + \left(\frac{b-a}{n}\right)^2\frac{n(n+1)}{2}\right) =$$

$$= \lim_{n\to\infty}\left(a(b-a) + \frac{(b-a)^2}{n}\frac{(n+1)}{2}\right) = a(b-a) + \frac{(b-a)^2}{2}\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = a(b-a) + \frac{(b-a)^2}{2} =$$

$$= \frac{2ab - 2a^2 + b^2 - 2ab + a^2}{2} = \frac{b^2 - a^2}{2}$$

Ex 5. $\int_a^b x^2\,dx = \lim_{n\to\infty}\sum_{j=1}^{n} x_j^2\Delta x = \lim_{n\to\infty}\sum_{j=1}^{n}(a+j\Delta x)^2\Delta x = \lim_{n\to\infty}\sum_{j=1}^{n}\left(a^2 + 2aj\Delta x + (j\Delta x)^2\right)\Delta x$

$$= \lim_{n\to\infty}\left(\sum_{j=1}^{n} a^2\Delta x + \sum_{j=1}^{n} 2aj(\Delta x)^2 + \sum_{j=1}^{n} j^2(\Delta x)^3\right) = \lim_{n\to\infty}\left(na^2\Delta x + 2a(\Delta x)^2\frac{n(n+1)}{2} + (\Delta x)^3\frac{n(n+1)(2n+1)}{6}\right) =$$

$$= a^2(b-a) + \lim_{n\to\infty}\left(a(b-a)^2\frac{(n+1)}{n} + \frac{(b-a)^3}{6}\frac{(n+1)(2n+1)}{n^2}\right) =$$

$$= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{6}\lim_{n\to\infty}\frac{(n+1)(2n+1)}{n^2}$$

$$= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{6}\lim_{n\to\infty}\frac{2n^2+3n+1}{n^2} = a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3} = \frac{b^3}{3} - \frac{a^3}{3}$$

Ex 6. $\int_a^b x^3\,dx = \lim_{n\to\infty}\sum_{j=1}^{n} x_j^3\Delta x = \lim_{n\to\infty}\sum_{j=1}^{n}(a+j\Delta x)^3\Delta x = \lim_{n\to\infty}\sum_{j=1}^{n}\left(a^3 + 3a^2 j\Delta x + 3a(j\Delta x)^2 + (j\Delta x)^3\right)\Delta x =$

$$= \lim_{n\to\infty}\sum_{j=1}^{n}\left(a^3\Delta x + 3a^2 j(\Delta x)^2 + 3aj^2(\Delta x)^3 + j^3(\Delta x)^4\right) =$$

$$= \lim_{n\to\infty}\left(a^3(b-a) + 3a^2\frac{(b-a)^2}{n^2}\frac{n(n+1)}{2} + 3a\frac{(b-a)^3}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{(b-a)^4}{n^4}\left(\frac{n(n+1)}{2}\right)^2\right) =$$

$$= \lim_{n\to\infty}\left(a^3(b-a) + \frac{3}{2}a^2(b-a)^2\frac{n+1}{n} + \frac{a(b-a)^3}{2}\frac{(n+1)(2n+1)}{n^2} + \frac{(b-a)^4}{4}\left(\frac{n+1}{n}\right)^2\right) =$$

$$= a^3(b-a) + \frac{3}{2}a^2(b-a)^2 + a(b-a)^3 + \frac{(b-a)^4}{4} = \frac{b^4}{4} - \frac{a^4}{4}$$

12.2.2 Midpoint Rule

It is sometime usable to estimate integral numerically. Instead of evaluating limits one choose finite (small) n and $x_j^* = \frac{x_{j-1}+x_j}{2}$, then use the definition.

Ex 7. The exact solution, as shown in previous example is $\frac{2^4}{4} = 4$

$$\int_0^2 x^3\,dx = \frac{1}{2}\left(\frac{0+1/2}{2}\right)^3 + \frac{1}{2}\left(\frac{1/2+1}{2}\right)^3 + \frac{1}{2}\left(\frac{1+3/2}{2}\right)^3 + \frac{1}{2}\left(\frac{3/2+2}{2}\right)^3 =$$

$$= \frac{1}{2}\left(\frac{1}{4}\right)^3 + \frac{1}{2}\left(\frac{3}{4}\right)^3 + \frac{1}{2}\left(\frac{5}{4}\right)^3 + \frac{1}{2}\left(\frac{7}{4}\right)^3 = \frac{1}{2}\left(\frac{1+27+125+343}{4^3}\right) = \frac{31}{8}$$

Ex 8. Approximate $\int_0^\pi \sin x\,dx = 2$ using Midpoint Rule, compare to "end point" rule (aka $x_j^* = x_j$).

- $\int_0^\pi \sin x\,dx \approx \pi \sin\frac{\pi}{2} = \pi$ $\qquad$ $\int_0^\pi \sin x\,dx \approx \pi\sin\pi = 0$
- $\int_0^\pi \sin x\,dx \approx \frac{\pi}{2}\sin\frac{\pi}{4} + \frac{\pi}{2}\sin\frac{3\pi}{4} = \frac{\pi}{2}\frac{1}{\sqrt{2}} + \frac{\pi}{2}\frac{1}{\sqrt{2}} = \frac{\pi}{\sqrt{2}} \approx 2.22$
  $\int_0^\pi \sin x\,dx \approx \frac{\pi}{2}\sin\frac{\pi}{2} + \frac{\pi}{2}\sin\pi = \frac{\pi}{2} + 0 = \frac{\pi}{2} \approx 1.5708$

12.2.3 Properties of Definite Integral

1. $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{j=1}^n f(x_j)\Delta x = \lim_{n\to\infty}\sum_{j=1}^n f(x_j)\frac{b-a}{n} = -\lim_{n\to\infty}\sum_{j=1}^n f(x_j)\frac{a-b}{n} = -\int_b^a f(x)dx$
2. $\int_a^a f(x)dx = \lim_{n\to\infty}\sum_{j=1}^n f(x_j)\frac{a-a}{n} = 0$
3. $\int_a^b c\,dx = \lim_{n\to\infty}\sum_{j=1}^n cf(x_j)\Delta x = c\lim_{n\to\infty}\sum_{j=1}^n f(x_j)\Delta x = c\int_a^b dx = c(b-a)$
4. $\int_a^b \{f(x)\pm g(x)\}dx = \lim_{n\to\infty}\sum_{j=1}^n \{f(x_j)\pm g(x_j)\}\Delta x =$
   $= \lim_{n\to\infty}\sum_{j=1}^n f(x_j)\Delta x \pm \lim_{n\to\infty}\sum_{j=1}^n g(x_j)\Delta x = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. $\int_a^b cf(x)dx = \lim_{n\to\infty}\sum_{j=1}^n cf(x_j)\Delta x = c\lim_{n\to\infty}\sum_{j=1}^n f(x_j)\Delta x = c\int_a^b f(x)dx$

6. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Ex 9.
$$\int_a^c dx + \int_c^b dx = (c-a)+(b-c) = b-a = \int_a^b dx$$
$$\int_a^c x\,dx + \int_c^b x\,dx = \left(\frac{c^2}{2}-\frac{a^2}{2}\right)+\left(\frac{b^2}{2}-\frac{c^2}{2}\right) = \frac{b^2}{2}-\frac{a^2}{2} = \int_a^b x\,dx$$

7. If $f(x) \ge 0, \forall x \in [a,b] \Rightarrow \int_a^b f(x)dx \ge 0$

8.
$$f(x) \ge g(x), \forall x \in [a,b] \Rightarrow f(x)-g(x) \ge 0$$
$$\Rightarrow \int_a^b f(x)-g(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx \ge 0 \Rightarrow \int_a^b f(x)dx \ge \int_a^b g(x)dx$$

9. $m \le f(x) \le M \Rightarrow \int_a^b m\,dx \le \int_a^b f(x)dx \le \int_a^b M\,dx \Rightarrow m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

Ex 10. Estimate $\int_0^1 x^3 - x^2 + 1\,dx = \frac{11}{12} \approx 0.916$: We would like to bound the function, i.e. we first find global min and max value of the integrand. Let $f(x) = x^3 - x^2 + 1$, the first derivative test give us $f'(x) = 3x^2 - 2x = 0 \Leftrightarrow x = 0, \frac{2}{3}$, we next compare value of *f* at these and end points. $f(0)=1;\ f(1)=1;\ f\left(\frac{2}{3}\right) = \frac{23}{27} \approx 0.852$. Thus

$$0.852 \le f(x) \le 1 \Rightarrow 1\cdot 0.852 \le \int_0^1 f(x)dx \le 1\cdot 1$$

12.3 Evaluating Definite Integrals (5.3)

Evaluation theorem: Let *f* be continuous on [a,b] and let *F* be arbitrary antiderivative of *f*, i.e. *F'=f*, then $\int_a^b f(x)dx = F(b) - F(a)$

In order to see why the theorem above works we consider dividing [a,b] into n subintervals with the following end points $a = x_0, x_1, ..., x_n = b$ and $\Delta x = x_{j+1} - x_j = \frac{b-a}{n}$. We next use Mean Value Theorem to see that $\frac{F(x_j) - F(x_{j-1})}{x_j - x_{j-1}} = f(x_j^*)$ where $x_j^* \in [x_{j-1}, x_j]$, thus $F(x_j)-F(x_{j-1})=f(x_j^*)\Delta x$. Therefore:

$$F(b)-F(a)=F(x_n)-F(x_0)=F(x_n)-F(x_{n-1})+F(x_{n-1})+\ldots+-F(x_1)+F(x_1)-F(x_0)=$$

$$=f(x_n^*)\Delta x+\ldots+f(x_1^*)\Delta x=\sum_{j=1}^{n} f(x_j^*)\Delta x$$

Note: $F(b)-F(a)=\lim\limits_{n\to\infty} F(b)-F(a)=\lim\limits_{n\to\infty}\sum_{j=1}^{n} f(x_j^*)\Delta x=\int_a^b f(x)dx$.

Notation: $\int_a^b f(x)dx=F(x)\big|_a^b=F(b)-F(a)$

Ex 11. $\int_a^b c\,dx=cx\big|_a^b=(b-a)c$

Ex 12. $\int_a^b x\,dx=\frac{1}{2}x^2\Big|_a^b=\frac{1}{2}(b^2-a^2)$

Ex 13. $\int_{-1}^{1} x^n dx=\frac{1}{n+1}x^{n+1}\Big|_{-1}^{1}=\frac{1}{n+1}\left(1-(-1)^{n+1}\right)=\frac{1}{n+1}\left(1+(-1)^n\right)=\begin{cases}2 & n=2m\\ 0 & n=2m+1\end{cases}$

Ex 14. $\int_0^{\pi}\sin x\,dx=-\cos x\big|_0^{\pi}=(-\cos\pi)-(-\cos 0)=1-\cos\pi=1-(-1)=2$

Ex 15.

$$\int_0^{\ln\pi}\sqrt{\left(e^t\cos t-e^t\sin t\right)^2+\left(e^t\sin t+e^t\cos t\right)^2}\,dt=\int_0^{\ln\pi}e^t\sqrt{(\cos t-\sin t)^2+(\sin t+\cos t)^2}\,dt=$$

$$=\int_0^{\ln\pi}e^t\sqrt{\left(\cos^2 t-2\cos t\sin t+\sin^2 t\right)+\left(\cos^2 t+2\cos t\sin t+\sin^2 t\right)}\,dt=$$

$$=\int_0^{\ln\pi}\sqrt{2e^{2t}}\,dt=\sqrt{2}\int_0^{\ln\pi}e^t dt=\sqrt{2}e^t\Big|_0^{\ln\pi}=\sqrt{2}(\pi-1)$$

12.3.1 Indefinite Integral

Def: The common notation of antiderivative is $\int f(x)dx$

$$\int f(x)dx=F(x) \text{ means } F'(x)=f(x)$$

Note that the difference between indefinite integral $\int f(x)dx$ and definite integral $\int_a^b f(x)dx$ is that the former is a function whereas the later is a number. The connection is give by Evaluation theorem: $\int_a^b f(x)dx=\int f(x)dx\Big|_a^b$

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Table of indefinite integrals:

1. $\int cf(x)dx = c\int f(x)dx$
2. $\int f(x)+g(x)dx = \int f(x)dx + \int g(x)dx$
3. $\int x^n\,dx = \frac{1}{n+1}x^{n+1}+C$
4. $\int \frac{1}{x}dx = \ln|x|+C$
5. $\int e^x\,dx = e^x+C$
6. $\int a^x\,dx = \frac{a^x}{\ln a}+C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sin x\,dx = -\cos x + C$
9. $\int \frac{1}{\cos^2 x}dx = \tan x + C$
10. $\int \frac{1}{\sin^2 x}dx = \cot x + C$
11. $\int \frac{1}{x^2+1}dx = \tan^{-1} x + C$
12. $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$

Ex 1. $\int (2x+7)\,dx = 2\int x\,dx + \int 7\,dx = \cancel{2}\frac{\cancel{1}}{\cancel{2}}x^2+7x+C$

Ex 2. $\int (3x+5)^2\,dx = \int (9x^2+30x+25)\,dx = 3\int 3x\,dx + 15\int 2x\,dx + 25\int dx =$
$= 3x^3+15x^2+25x+C$

Ex 3. $\int \frac{dx}{1+\cos 2x} = \int \frac{dx}{2\cos^2 x} = \frac{1}{2}\tan x + C$

Ex 4. $\int \sin^2 x dx = \int \frac{1-\cos(2x)}{2}dx = \frac{x}{2}-\frac{\sin(2x)}{4}+C$

Ex 5. $\int \sin 2x \cos 3x dx = \frac{1}{2}\int \sin(-x)+\sin(5x)dx = \frac{\cos x}{2}-\frac{\cos(5x)}{10}+C$

Ex 6. $\int \tan^2 t = \int 1+\frac{1}{\cos^2 t} = t+\tan t + C$