12.4 The Fundamental Theorem Calculus (5.4)

Fundamental Theorem of Calculus provides a connection between differentiable and integral calculus.

Consider we interesting in computing the area that lies between a positive function $f(x)$ and x-axis on [a,b], i.e. $A=\int_a^b f(t)\,dt$. In additional let the upper limit b vary, i.e. we consider a function $A(x)=\int_a^x f(t)\,dt$.

Ex 1. $\int_a^x \cos(t)\,dt = \sin t\big|_a^x = \sin x - \sin a$

Ex 2. $\int_0^x t^3\,dt = \frac{1}{4}t^3\Big|_0^x = \frac{x^3-0}{4}$

Let F be antiderivative of f, calculate the derivative of $A(x)=\int_a^x f(t)\,dt$:

$$\frac{d}{dx}A(x)=\frac{d}{dx}\int_a^x f(t)\,dt=\frac{d}{dx}\big(F(x)-F(a)\big)=F'(x)=f(x)$$

Thus $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Ex 3. Does $f(x)=\begin{cases}0 & 0\le x<1\\ 1 & 1\le x\le 2\end{cases}$ has antiderivative?

Suppose $F(x)$ is an antiderivative of $f(x)$ on [0,2], then $F(x)$ is differentiable function, furthermore $F'(x)=f(x)$. Define $g(x)=F(x)-Cx$ for an arbitrary constant $C\in\big(f(0),f(2)\big)=(0,1)$. The function g is clearly continuous function on [0,2], because it differentiable, and therefore there is c, s.t. g'(c)=0. But g'(c)=f(c)-C, i.e. f(c)=C for an arbitrary $C\in(0,1)$ which is impossible. Note that $f(0)<C<f(2)$ so $c\ne 0$ and $c\ne 2$. Thus f has no antiderivative.

Let calculate $\frac{d}{dx}\int_a^x f(t)\,dt$ for continuous f without assumptions of antiderivative existence:

$$\frac{d}{dx}\int_a^x f(t)dt = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$$

$$\int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$$

$$\Rightarrow \frac{d}{dx}\int_a^x f(t)dt = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt$$

From the continuity of $f(x)$ on [x,x+h] we have

$$f(x_m) = m < f(x) < M = f(x_M) \Rightarrow mh < \int_x^{x+h} f(t)dt < Mh \Rightarrow m < \frac{1}{h}\int_x^{x+h} f(t)dt < M .$$

Since $x_m, x_M \in [x, x+h]$ and $h \to 0$ we have $x_m, x_M \to x$, so $f(x_m) = m, f(x_M) = M \to f(x)$, therefore by intermediate value theorem $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

The Fundamental Theorem of Calculus:

Suppose f(x) is continuous on [a,b].

1) If $g(x) = \int_a^x f(t)dt$ then $g'(x) = f(x)$
2) $\int_a^b f(t)dt = F(b) - F(a)$ for any F antiderivative of f, that is $F' = f$

Ex 4. $\frac{d}{dx}\int_0^x \cos^2 t\, dt = \cos^2 x = \cos^2 x$

Ex 5. $\frac{d}{dx}\int_3^{1/x} \sqrt{x}\, dx = \frac{1}{\sqrt{x}}\cdot\left(-\frac{1}{x^2}\right)$

Ex 6.

$$\frac{d}{dx}\int_{\sin^2 x}^{x^2} \frac{1}{t}dx = \frac{d}{dt}\left(\int_{\sin^2 x}^{c} \frac{1}{t}dt + \int_c^{x^2}\frac{1}{t}dt\right) = \left(-\frac{d}{dx}\int_c^{\sin^2 x}\frac{1}{t}dt + \frac{d}{dx}\int_c^{x^2}\frac{1}{t}dx\right) =$$

$$= -\frac{1}{\sin^2 x}\cdot(2\sin x\cos x) + \frac{1}{x^2}\cdot 2x = 2\left(\frac{1}{x} - \cot x\right)$$

12.5The Substitution Rule (5.5)

Indefinite Integral: $\int_a^b f(g(x))g'(x)dx \underset{\substack{t=g(x)\\ dt=g'(x)dx}}{=} \int f(t)dt$

Ex 1. $\int (2x+7)\,dx \underset{\substack{t=2x+7\\dt=2dx}}{=} \int \frac{1}{2}t\,dt = \frac{1}{4}t^2 = \frac{1}{4}(2x+7)^2 + \tilde{C} = x^2 + 7x + \underbrace{\frac{49}{4} + \tilde{C}}_{=C}$

Ex 2. $\int (3x+5)^2\,dx \underset{\substack{t=3x+5\\dt=3dx}}{=} \int \frac{1}{3}t^2\,dt = \frac{1}{3}\frac{t^3}{3} + C = \frac{1}{9}(3x+5)^3 + C = 3x^3 + 15x^2 + 25x + \tilde{C}$

Ex 3. $\int \frac{x}{\sqrt{3x+5}}dx \underset{\substack{t=x+5/3\\dt=dx}}{=} \frac{1}{\sqrt{3}}\int \frac{t-\frac{5}{3}}{\sqrt{t}}dt = \frac{1}{\sqrt{3}}\int \sqrt{t} - \frac{5}{3}t^{-1/2} = \frac{1}{\sqrt{3}}\left(\frac{2}{3}t^{\frac{3}{2}} - \frac{5}{6}\sqrt{t}\right)$

Ex 4. $\int \frac{dx}{1+\cos x} = \int \frac{dx/2}{\cos^2(x/2)} \underset{\substack{t=\frac{x}{2}\\dt=dx/2}}{=} \int \frac{dt}{\cos^2 t} = \tan t + C = \tan\left(\frac{x}{2}\right) + C$

Ex 5. $\int u^n(x)u'(x)dx = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1$

Ex 6. $\int \frac{\tan^2 x}{\cos^2 x}dx \underset{\substack{u=\tan x\\du=dx/\cos^2 x}}{=} \int u^2\,du = \frac{1}{3}u^3 = \frac{1}{3}\tan^3 x$

Ex 7. $\int \frac{dx}{x} = \ln|x| + C \qquad \int \frac{u'(x)}{u(x)}dx = \ln|u| + C$

Ex 8. Show that if $\int \frac{f'(x)}{f(x)+1}dx = -x^3$ then $f(x) = 0, \forall x \geq 0$

Rewrite it as $\ln(f(x)+1) = -x^3$ and learn that f(x)+1>0. Next, implicitly differentiate $\int \frac{f'(x)}{f(x)+1}dx = -x^3$ to get $\frac{f'(x)}{f(x)+1} = -2x^2$ which gives $f' = -2x^2(f(x)+1) < 0$, i.e. f(x) is decreasing function. However $x = 0 \Rightarrow \ln(f(0)+1) = -0^3 = 0$, thus $f(0)+1 = 1$ and we get $f(0) = 0$. Since f decreasing starting from 0 we get $f(x) = 0, \forall x \geq 0$.

Ex 9. $\int \frac{x^4}{x^5-9}dx = \frac{1}{5}\int \frac{5x^4}{x^5-9}dx = \frac{1}{5}\int \frac{dx^5}{x^5-9} = \frac{1}{5}\ln|x^5-9| + C$

Ex 10. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx \underset{\substack{u=\cos x\\du=-\sin x dx}}{=} \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C$

Inverse Substitution

Ex 11. $$\int \frac{dx}{\sqrt{x}\left(1+\sqrt[3]{x}\right)} \underset{\substack{x=t^6\\ dx=6t^5dt}}{\overset{x>0}{=}} \int \frac{6t^5dx}{t^3\left(1+t^2\right)} = 6\int \frac{t^2dx}{1+t^2} = 6\int\left(1-\frac{1}{1+t^2}\right)dx =$$
$$= 6\left(t-\arctan t\right)+C = 6\left(\sqrt[6]{x}-\arctan\sqrt[6]{x}\right)+C$$

$$\int \sqrt{1-x^2}\,dx \underset{\substack{x=\sin t\\ dx=\cos t dt}}{=} \int \sqrt{1-\sin^2 t}\cos t\,dt = \int \sqrt{\cos^2 t}\cos t dt = \int \cos^2 t\,dt =$$
$$= \int \frac{1+\cos(2t)}{2}dt = \frac{t}{2}+\frac{\sin(2t)}{4}+C = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right)+C$$
$$\sin 2t = 2\sin t\cos t = 2\sin t\sqrt{1-\sin^2 t}$$

Definite Integral Substitution: $\int_a^b f\left(g(x)\right)g'(x)\,dx \underset{\substack{t=g(x)\\ dt=g'(x)dx}}{=} \int_{g(a)}^{g(b)} f(t)dt$

$$\int_0^\pi \cos x\sin x\,dx \underset{\substack{t=\cos x\\ dt=-\sin x\ dx\\ x=0\to t=1\\ x=\pi\to t=-1}}{=} -\int_{-1}^{1} t\,dt = -\frac{t^2}{2}\bigg|_{-1}^{1} = -\frac{1}{2}\left(1^2-(-1)^2\right)=0$$

Ex 12. $$\int_0^\pi \cos x\sin x\,dx = \int \cos x\sin x\,dx\bigg|_{x=0}^{x=\pi} \underset{\substack{t=\cos x\\ dt=-\sin x\ dx}}{=} -\int t\,dt\bigg|_{x=0}^{x=\pi} = -\frac{t^2}{2}\bigg|_{x=0}^{x=\pi} = -\frac{\cos^2 x}{2}\bigg|_0^\pi =$$
$$= -\frac{1}{2}\left(\cos^2\pi-\cos^2 0\right) = -\frac{1}{2}\left(1^2-(-1)^2\right)=0$$

Ex 13. $$\int_0^1 \sqrt{6-5x} \underset{\substack{t=6-5x\\ dt=-5dx}}{=} \int_6^1 \sqrt{t}\frac{dt}{-5} = \frac{1}{5}\int_1^6 \sqrt{t}\,dt = \frac{1}{5}\frac{2}{3}t^{3/2}\bigg|_1^6 = \frac{2}{15}\left(6^{3/2}-1^{3/2}\right)$$

Integral of Symmetric Function:

If f is even

$$\int_{-a}^0 f(x)dx \underset{\substack{t=-x\\ dt=-dx\\ (-a,0)\to(a,0)}}{=} -\int_a^0 f(-x)dx = \int_0^a f(-x)dx \underset{f(x)=f(-x)}{=} \int_0^a f(x)dx$$
$$\Rightarrow \int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$$

If f is odd then

$$\int_{-a}^{0} f(x)\,dx \underset{\substack{t=-x \\ dt=-dx \\ (-a,0)\to(a,0)}}{=} -\int_{a}^{0} f(-x)\,dx = \int_{0}^{a} f(-x)\,dx \underset{-f(x)=f(-x)}{=} -\int_{0}^{a} f(x)\,dx$$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

12.6 Integration be parts (5.6)

Recall the Chain Rule $\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$, rewrite in a different way $d(fg) = gdf + fdg$ and then integrate (both sides) $\int d(fg) = fg = \int g\,df + \int f\,dg$. Thus $\int f\,dg = fg - \int g\,df$

Ex 1. $\int x\sin x dx = -\int x\cos' x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$

Ex 2. $8\int x\sin x\cos x = 4\int x\sin 2x dx \underset{\substack{t=2x \\ dt=2dx}}{=} \int t\sin t dt = -t\cos t + \int \cos t\,dt$

$= -t\cos t + \sin t + C = -2x\cos 2x + \sin 2x + C$

Ex 3. $\int 1\cdot \ln x\,dx = x\cdot\ln x - \int dx = x\cdot\ln x - x + C$

Ex 4. $\int 1\cdot\arctan x\,dx = x\cdot\arctan x - \int \frac{x}{x^2+1}dx = x\cdot\arctan x - \frac{1}{2}\int\frac{2x}{x^2+1}dx =$

$= x\cdot\arctan x - \frac{1}{2}\int \frac{(x^2+1)'}{x^2+1}dx = x\cdot\arctan x - \frac{1}{2}\ln(1+x^2) + C$

Definite Integral: $\int_a^b f\,dg = fg\big|_a^b - \int_a^b g\,df$

Ex 5. $\int_0^{\pi} x\cos x\,dx = \int_0^{\pi} x\sin' x\,dx = \underbrace{x\sin x\big|_0^{\pi}}_{=0} - \int_0^{\pi}\sin x\,dx = \cos x\big|_0^{\pi} = \cos\pi - \cos 0 = -1 - 1 = -2$