12.7 Additional Techniques of Integration (5.7)

Ex 1.
$$\int t^2\sin t\,dt = \int t^2\sin t\,dt = -t^2\cos t - 2\int t(-\cos t)dt = -t^2\cos t + 2\int t\cos t\,dt =$$
$$= -t^2\cos t + 2t\sin t - 2\int \sin t\,dt = -t^2\cos t + 2t\sin t + 2\cos t = \left(2-t^2\right)\cos t + 2t\sin t$$

Ex 2. Find $\int \cos^3 x dx$:

$$I_3 = \int \cos^3 x\,dx = \int \cos^2 x\cos x\,dx = \int \cos^2 x\sin' x\,dx = \cos^2 x\sin x + \int 2\cos x\sin^2 x\,dx =$$
$$= \cos^2 x\sin x + \int 2\cos x\left(1-\cos^2 x\right)dx = \cos^2 x\sin x + 2I_1 - 2I_3$$
$$\Rightarrow I_3 = \frac{1}{3}\left(\cos^2 x\sin x + 2I_1\right), I_1 = \int \cos x\,dx = \sin x + C$$
$$\Rightarrow I_3 = \frac{1}{3}\left(\cos^2 x\sin x + 2\sin x\right) + C$$

Note: $I_n = \frac{1}{n}\left(\cos^{n-1} x\sin x + (n-1)I_{n-2}\right)$

12.7.1 Trigonometric Substitution:

12.7.1.1 The substitution $t=\tan\frac{x}{2}$ is useful for sum/diff/mult/fraction of sin and cos:

In this case $dx = \frac{2}{1+t^2}dt; \cos x = \frac{1-t^2}{1+t^2}; \sin x = \frac{2t}{1+t^2}$

Ex 3.
$$\int \frac{dx}{\sin x} \underset{t=\tan x/2}{=} \int \frac{1+t^2}{2t}\frac{2}{1+t^2}dt = \int \frac{1}{t}dt = \ln|t| + C = \ln\left|\tan\frac{x}{2}\right| + C$$

Ex 4.
$$\int \frac{dx}{\cos x} \underset{t=\tan x/2}{=} \int \frac{\cancel{1+t^2}}{1-t^2}\frac{2}{\cancel{1+t^2}}dt = \int \frac{1}{1+t} + \frac{1}{1-t}dt =$$
$$= \ln|1+t| - \ln|1-t| + C = \ln\left|\frac{1+\tan x/2}{1-\tan x/2}\right| + C$$

Ex 5. For $a > |b|$

$$\int \frac{dx}{a+b\cos x} \underset{\substack{t=\tan(x/2) \\ dx=\frac{2}{1+t^2}dt \\ \cos x=\frac{1-t^2}{1+t^2}}}{=} \int \frac{\frac{2}{1+t^2}dt}{a+b\frac{1-t^2}{1+t^2}} = \int \frac{2}{a\left(1+t^2\right)+b\left(1-t^2\right)}dt = \int \frac{2}{a+b+(a-b)t^2}dt =$$

$$= \frac{2}{a+b}\int \frac{1}{1+\frac{a-b}{a+b}t^2}dt \underset{\substack{y=t\sqrt{\frac{a-b}{a+b}} \\ dy=\sqrt{\frac{a-b}{a+b}}dt}}{=} \frac{2}{a+b}\sqrt{\frac{a+b}{a-b}}\int \frac{1}{1+y^2}dy = \frac{2\arctan y}{\sqrt{(a+b)(a-b)}} =$$

$$= \frac{2}{\sqrt{(a+b)(a-b)}}\arctan t\sqrt{\frac{a-b}{a+b}} = \frac{2}{\sqrt{(a+b)(a-b)}}\arctan\left(\tan\frac{x}{2}\sqrt{\frac{a-b}{a+b}}\right)$$

12.7.1.2 Substitution of the form $\int f(\cos)\sin; \int f(\sin)\cos$

Ex 6.

$$\int \sin x\cdot\cos x\cdot dx \underset{\substack{u=\sin x \\ du=\cos x dx}}{=} \int u\cdot du = \frac{1}{2}u^2+C = \frac{1}{2}\sin^2 x+C$$

Ex 7.

$$\int_0^{\pi/2} \cos x\sqrt{1-\cos(\sin x)}\,dx \underset{\substack{t=\sin x \\ dt=\cos x dx}}{=} \int_0^1 \sqrt{1-\cos t}\,dt = \int_0^1 \sqrt{2\sin^2\frac{t}{2}}\,dt \underset{\sin\frac{t}{2}>0}{=} \sqrt{2}\int_0^1 \sin\frac{t}{2}dt =$$

$$\underset{\substack{u=t/2 \\ du=dt/2}}{=} 2^{3/2}\int_0^{1/2}\sin u\,du = 2^{3/2}\left(-\cos u\right)_0^{1/2} = -2^{3/2}\left(\cos\frac{1}{2}-1\right) = 2^{3/2}\left(1-\cos\frac{1}{2}\right) = 2^{5/2}\sin^2\frac{1}{4}$$

12.7.1.3 Substitution $t=\tan x$ useful for sum/diff/mult/fraction of tan and even powers of sin and cos

In this case $\sin x = \frac{t}{\sqrt{1+t^2}};\ \cos x = \frac{1}{\sqrt{1+t^2}};\ dt = \frac{1}{\cos^2 x} = \left(1+\tan^2 x\right) = \frac{1}{1+t^2}$

Ex 8.

$$\int \tan^3 x\,dx \underset{\substack{t=\tan x \\ dt=\frac{dx}{\cos^2 x}=\left(1+\tan^2 x\right)dx}}{=} \int \frac{t^3dt}{1+t^2} = \int t-\frac{t}{1+t^2}dt = \frac{1}{2}\left(t^2-\ln\left(t^2+1\right)\right)+C =$$

$$= \frac{1}{2}\left(\tan^2 x-\ln\left(\tan^2 x+1\right)\right)+C$$

Ex 9.

$$\int \frac{dx}{\sin^4 x\cos^2 x} \underset{\substack{t=\tan x \\ dt=dx/\cos^2 x}}{=} \int\left(1+\frac{1}{t^2}\right)^2 dt = \int\left(1+\frac{2}{t^2}+\frac{1}{t^4}\right)dt = t-\frac{2}{t}-\frac{1}{3t^3}+C =$$

$$= \tan x - 2\cot x + \frac{1}{3}\cot^3 x$$

$$\frac{1}{\sin^4 x} = \left(\frac{\sin^2 x+\cos^2 x}{\sin^2 x}\right)^2 = \left(1+\frac{1}{\tan^2 x}\right)^2$$

12.7.2 Partial Fractions:

$$\int \frac{adx}{(x-b)^k} = a\int \frac{dt}{t^k} = C + \begin{cases} a\ln|t| & k=1 \\ a\dfrac{t^{1-k}}{1-k} & k\ge 2 \end{cases} = C + \begin{cases} a\ln|x-b| & k=1 \\ a\dfrac{(x-b)^{1-k}}{1-k} & k\ge 2 \end{cases}$$

Ex 10. $\int \frac{x^2-3x+2}{x^3+2x^2+x}dx$

$$\frac{x^2-3x+2}{x(x+1)^2} = \frac{a}{x} + \frac{b}{(x+1)} + \frac{c}{(x+1)^2} = \frac{a(x+1)^2 + b(x+1)x + cx}{x(x+1)^2}$$

$$a(x+1)^2 + b(x+1)x + cx = ax^2 + 2ax + a + bx^2 + bx + cx =$$

$$= (a+b)x^2 + (b+c+2a)x + a = x^2 - 3x + 2 \Rightarrow a = 2; a+b = 1; b+c+2a = -3\ldots$$

$$\int \frac{x^2-3x+2}{x^3+2x^2+x}dx = \int \frac{x^2-3x+2}{x(x^2+2x+1)}dx = \int \frac{x^2-3x+2}{x(x^2+2x+1)}dx = \int \frac{x^2-3x+2}{x(x+1)^2}dx \underset{*}{=}$$

$$= \int \left(\frac{2}{x} - \frac{1}{x+1} - \frac{6}{(x+1)^2}\right)dx = 2\ln x - \ln(x+1) + \frac{6}{x+1} + C$$

Ex 11. $\int \frac{12s+11}{(s+4)^2(s^2-4s+5)}ds$

$$\frac{12s+11}{(s+4)^2(s^2-4s+5)} = \frac{A}{(s+4)} + \frac{B}{(s+4)^2} + \frac{Cs+D}{(s^2-4s+5)} = \frac{A(s+4)(s^2-4s+5) + B(s^2-4s+5) + (Cs+D)(s+4)^2}{(s+4)^2(s^2-4s+5)} =$$

$$= \frac{(A+C)s^3 + (B+8C+D)s^2 + (8D-11A+16C-4B)s + (20A+5B+16D)}{(s+4)^2(s^2-4s+5)}$$

we get $\begin{cases} A+C=0 \\ B+8C+D=0 \\ 8D-11A+16C-4B=12 \\ 20A+5B+16D=11 \end{cases} \Rightarrow A=0, B=-1, C=0, D=1$

$$\int \frac{12s+11}{(s+4)^2(s^2-4s+5)}ds = -\int \frac{ds}{(s+4)^2} + \int \frac{ds}{s^2-4s+5} = \frac{1}{4+s} + \int \frac{ds}{(s-2)^2+1} = \frac{1}{4+s} + \arctan(s-2)$$

(completing the square: $s^2-4s+5 = (s-2)^2+1$)

More general case:

$$\int \frac{ax+b}{\underbrace{\left(x^2+px+q\right)}_{no\,real\,roots}{}^k}dx = \int \frac{\frac{a}{2}\left(2x+p\right)+b-\frac{ap}{2}}{\left(x^2+px+q\right)^k}dx = \frac{a}{2}\int \frac{\overbrace{2x+p}^{dt}}{\underbrace{\left(x^2+px+q\right)}_{t}{}^k}dx +$$

$$+\left(b-\frac{ap}{2}\right)\int\left(\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right)^{-k}dx = \frac{a}{2}\int\frac{dt}{t^k}dx+\left(b-\frac{ap}{2}\right)\int\frac{dx}{\left(t^2+\alpha^2\right)^k}$$

$$I_k = \int\frac{dx}{\left(t^2+\alpha^2\right)^k} \Rightarrow I_{k+1} = \frac{1}{2k\alpha^2}\left(\frac{t}{\left(t^2+\alpha^2\right)^2}+\left(2k-1\right)I_k\right); I_1 = \frac{1}{\alpha}\arctan\frac{t}{\alpha}+C$$

12.8 Integration using tables and Computer Algebra System (5.8)

Self-reading.