

12.11.5 Applications (6.6-6.7)

Here we provide few more applications of integration.

12.11.5.1 Work

Consider Force = mass \times acceleration, i.e. let $s(t)$ be position function, then

$F = m \cdot s''(t)$. For a constant acceleration force is constant and then Work=force \times

distance, i.e. $W = F \cdot d$. When a force isn't constant, and a continuous force $f(x)$ act on the object along interval $[a,b]$, then **the work done in moving the object from a to b**

is given by $W = \lim_{n \rightarrow \infty} \sum_{j=1}^n f(x_j^*) \Delta x = \int_a^b f(x) dx$.

Ex 2. A force of 40N is required to hold a spring that has been stretched from it's natural length of 10 cm to a length of 15 cm. how much work is done in stretching the spring from 15 cm to 18 cm.

By Hooke's law force is given by $f(x) = kx$, where x is a displacement from the natural length of spring given in meters. Thus $0.05k = 40\text{N}$ and the spring constant

$k = \frac{40}{0.05} = 800$. Therefore we have $f(x) = 800x$ and the work is

$$W = \int_{0.05}^{0.08} 800x = 800 \left. \frac{x^2}{2} \right|_{0.05}^{0.08} = 1.56 J.$$

12.11.5.2 Hydrostatic Force and Pressure

Consider the horizontal plate with area of $A \text{ m}^2$ is submerged in a fluid of density $\rho \text{ kg/m}^3$ at depth d meter below the surface of the fluid. The fluid directly above the place has volume $V = Ad$, which means it has mass of $m = \rho V$ and force

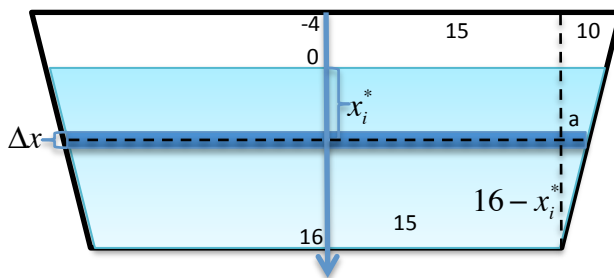
$F = mg = \rho \cdot Ad \cdot g$, where g is gravity (~ 9.8). A pressure, which is measured in

(kilo)Pascals, on this plate is given by $P = \frac{F}{A} \text{ N/m}^2 = \frac{F}{A} \text{ Pa} = \rho g d \text{ Pa}$. It is known that at

given depth the pressure is the same in all directions. This fact helps to determine a force against a **vertical** plate\wall\dam, which isn't straightforward since the pressure vary with the depth.

Ex 3. A trapezoidal dam have a height 20m, it's width is 50m at top and 30m at bottom. Find a force on the dam when the water level is 4m from the top.

In order to find the force we will use the formula $F = PA$. The first thing we need is to find the area at each depth (x_j^*). Thus, we consider the level of the water from the top to the bottom, thus the origin ($x=0$) is at the top and the bottom $x=20-4=16$. We divide the $[0,16]$ interval to small subintervals and measure the area of resulting rectangles between x_j and x_{j+1} which height is Δx .



The width of the rectangular is given by $2(15+a)$, where a satisfy $\frac{a}{16-x_j^*} = \frac{10}{20}$,

because of similar triangles (see the figure) Thus, $a = \frac{1}{2}(16-x_j^*)$ and the area is

$$A_j = 2(15+a)\Delta x = 2\left(15 + \frac{1}{2}(16-x_j^*)\right)\Delta x = (30+16-x_j^*)\Delta x = (46-x_j^*)\Delta x.$$

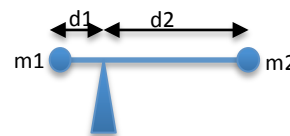
The density of the water is $\rho = 1000 \text{ kg/m}^3$, so the pressure is $P \approx 1000gx_j^* \approx 9800x_j^*$ and the force is

$$F = \lim_{n \rightarrow \infty} \sum_{j=1}^n P_j A_j = \lim_{n \rightarrow \infty} \sum_{j=1}^n 9800x_j^* (46-x_j^*) \Delta x = 9800x_j^* \int_0^{16} x(46-x) dx =$$

$$9800x_j^* \left(46 \frac{x^2}{2} - \frac{x^3}{3} \right)_0^{16} \approx 4.43 \times 10^7 \text{ N}$$

12.11.5.3 Moments and center of mass

Consider that a masses m_1 and m_2 attached to a rod of negligible mass on opposite sides of fulcrum. Let where d_1, d_2 be the distances of the masses from the fulcrum. Then, by the Law of the Lever, discovered by Archimedes, the rod will balance if $m_1 d_1 = m_2 d_2$. In such situation fulcrum is at **center of mass**.



If we locate the rod on the rule so the masses are at x_1, x_2 and the fulcrum at \bar{x} , the law of Lever become $m_1(\bar{x} - x_1) = m_2(\bar{x} - x_2)$. We now can find the center of mass by

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \text{ The numbers } m_1 x_1, m_2 x_2 \text{ called } \mathbf{moments}.$$

If we generalize this and put more weights along the rod we need to sum up the moments and masses in order to find the center of mass, i.e. $\bar{x} = \frac{\sum_{j=1}^n m_j x_j}{\sum_{j=1}^n m_j}$. We denote

$m = \sum_{j=1}^n m_j$ a total mass of the system and $M = \sum_{j=1}^n m_j x_j$ the moment of the system about the origin. We can rewrite it as following $m\bar{x} = M$, i.e. if all mass were concentrated at the center of mass, then its moment would be the same as the moment of the system.

Similarly one define a two dimensional center of mass by moment of the system.

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$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right) = \frac{1}{m} \left(\sum_{j=1}^n m_j x_j, \sum_{j=1}^n m_j y_j \right),$$

where M_x, M_y denotes the moments of the system about x and y axes respectively.

Ex 4. Find the moments and center of mass of the system of objects that have masses 1,5,7 at the points (0,-3), (1,3) and (3,2).

$$M_x = 1(0) + 5(1) + 7(3) = 0 + 5 + 21 = 26$$

$$M_y = 1(-3) + 5(3) + 7(2) = -3 + 15 + 14 = 26$$

$$m = 1 + 5 + 7 = 13$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_x}{m}, \frac{M_y}{m} \right) = \left(\frac{26}{13}, \frac{26}{13} \right) \approx (2, 2)$$

Consider a flat plate (lamina) with uniform density ρ . The center of mass of the plate is called centroid. The symmetry principle says that if the plate is symmetric about the some line, then the centroid lies on that line. Thus the centroid of rectangle is in its center.

Consider that we looking for a centroid of region enclosed by a,b x-axis and continuous function $f(x)$. We divide the interval $[a,b]$ into n subintervals of width Δx and choose $x_j^* = \bar{x}_j = \frac{1}{2}(x_{j-1} + x_j)$. Thus we get rectangles with centroids at $C_j(\bar{x}_j, \frac{1}{2}f(\bar{x}_j))$ and area $A_j = f(\bar{x}_j)\Delta x$, which mass is $m_j = \rho f(\bar{x}_j)\Delta x$.

The moment of rectangle j about the y-axis is it's mass times the distance from centroid to the y-axis, thus $M_y(R_j) = m_j \bar{x}_j = \bar{x}_j \rho f(\bar{x}_j) \Delta x$ and therefore

$$M_y = \lim_{n \rightarrow \infty} \sum_{j=1}^n M_y(R_j) = \lim_{n \rightarrow \infty} \sum_{j=1}^n \rho \bar{x}_j f(\bar{x}_j) \Delta x = \rho \int_a^b x f(x) dx$$

Similarly, the moment of rectangle j about the x-axis is it's mass times the distance from centroid to the x-axis, thus $M_x(R_j) = m_j \cdot \frac{1}{2} f(\bar{x}_j) = \rho f(\bar{x}_j) \Delta x \cdot \frac{1}{2} f(\bar{x}_j) = \frac{1}{2} \rho f^2(\bar{x}_j) \Delta x$ and therefore

$$M_x = \lim_{n \rightarrow \infty} \sum_{j=1}^n M_x(R_j) = \lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{1}{2} \rho f^2(\bar{x}_j) \Delta x = \frac{1}{2} \rho \int_a^b f^2(x) dx$$

The mass of the plate is given by $m = \rho A = \rho \int_a^b f(x) dx$, thus

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x) dx}{\rho \int_a^b f(x) dx} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx} = \frac{1}{A} \int_a^b x f(x) dx \quad \bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2} \rho \int_a^b f^2(x) dx}{\rho \int_a^b f(x) dx} = \frac{\frac{1}{2} \int_a^b f^2(x) dx}{\int_a^b f(x) dx} = \frac{1}{2A} \int_a^b f^2(x) dx$$

Ex 5. Find a center of mass of a half disc of radius r .

We define $f(x) = \sqrt{r^2 - x^2}$ on a $[-r, r]$ interval. The area is $A = \frac{1}{2} \pi r^2$. By symmetry principle we have $\bar{x} = 0$.

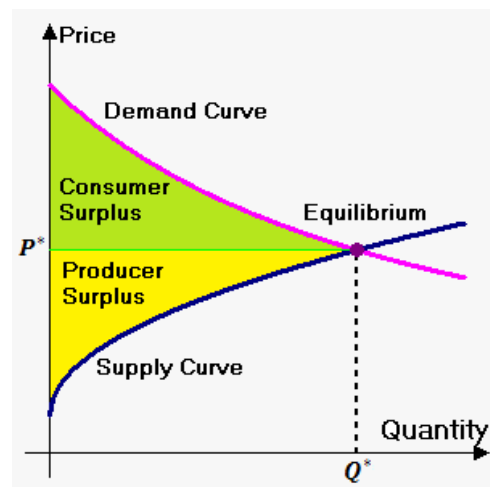
$$\begin{aligned} \bar{y} &= \frac{1}{2A} \int_a^b f^2(x) dx = \frac{1}{\pi r^2} \int_{-r}^r \left(\sqrt{r^2 - x^2} \right)^2 dx = \frac{2}{\pi r^2} \int_0^r r^2 - x^2 dx = \frac{2}{\pi r^2} \left(r^2 x - \frac{x^3}{3} \right)_0^r = \\ &= \frac{2}{\pi r^2} \left(r^3 - \frac{r^3}{3} \right) = \frac{4r}{3\pi} \approx 0.42r \end{aligned}$$

12.11.5.4 Consumer Surplus

The Supply function S represents the given quantity of a good that a supplier will produce at a given price. As price raises quantity supplied increases, thus S is increasing function.

The Demand function D represents the given quantity of a good that is demanded by consumers at a given price. The quantity demanded decreases as price rises, thus D is decreasing function.

The Supply and Demand Function intersect at the equilibrium price (p^*) and the equilibrium quantity (Q^*), i.e. $D(Q^*) = S(Q^*)$.



The revenue generated by Demand is the area under graph of D , i.e. $R = \int_0^n D(q) dq$.

In a competitive market consumers do not pay the price they are willing to pay for a good. They pay the equilibrium price (p^*) for the good. Therefore, the total revenue generated is p^*Q^* . In such situation some consumers pay less than they were willing to pay for the good. Such consumers experience consumer surplus. The consumer surplus can be computed using $S_{cus} = \int_0^{Q^*} D(q) - p^* dq$.

Similarly, a supplier experiences the producer surplus: $S_{sup} = \int_0^{Q^*} p^* - S(q) dq$.

12.11.5.5 Blood Flow

Recall the velocity of blood inside the vein formula given by The Law of Laminar (blood) Flow: $v = \frac{P}{4\eta l}(R^2 - r^2)$ where the vein is approximated by cylindrical tube of radius R and length l , r is the distance from the axis, P is the pressure difference between the ends of the tube and η is the viscosity of the blood. The rate of blood flow (flux) given in Volume/Time is calculated very similar to cylindrical shells idea, the radius interval $[0, R]$ is divided into r_1, r_2, \dots, r_n of the ring with inner/outer radius r_{j-1}/r_j respectively. The area of the ring is approximately $2\pi r_i \Delta r$ and therefore the volume is $2\pi r_i \Delta r \cdot v(r_i)$. Thus, the flux is called Poiseuille's Law:

$$F = \lim_{n \rightarrow \infty} \sum_{j=1}^n 2\pi r_i \cdot v(r_i) \Delta r = 2\pi \int_0^R r v(r) dr = 2\pi \int_0^R r \frac{P}{4\eta l} (R^2 - r^2) dr = \frac{P\pi}{2\eta l} \int_0^R r (R^2 - r^2) dr =$$

$$\frac{P\pi}{2\eta l} \left(R^2 \frac{r^2}{2} - \frac{r^4}{4} \right)_0^R = \frac{P\pi}{2\eta l} \left(\frac{R^4}{2} - \frac{R^4}{4} \right) = \frac{P\pi R^4}{8\eta l}$$

12.11.5.6 Cardiac Output

The cardiac output is the volume of blood pumped by the heart per time, i.e. the rate of flow. It is measured by injecting dye (a color) into the “entry”, a right atrium, and the concentration of leaving dye is measured at the “exit”, a probe is inserted into the aorta, at equally spaced times over time interval $[0, T]$. Let the concentration at time t be $c(t)$ then the amount of dye at the measuring point during subinterval (t_{i-1}, t_i) is concentration \cdot volume $= c(t_i)F\Delta t$, where the F is the rate of flow that we are trying to

determine. Thus the amount of dye is $A(T) = \lim_{n \rightarrow \infty} F \sum_{j=1}^n F c(t_j) \Delta t = F \int_0^T c(t) dt$ and we got

$$F = \frac{A}{\int_0^T c(t) dt}, \text{ where } A \text{ is known.}$$

12.11.6 Probability (6.8)

Probability is strongly connected with randomness. Therefore we start with several definitions. What will be a height of randomly chosen person? We have no single answer for this question, we rather can provide a range of possible values or an average value, for example in 2003-2006 American men of 20-29 ages had height of 1.776m while similar women 1.632m. Such quantities are called **random variables (RV)**; we will use continuous RV, CRV.

What is the chance (probability) that the randomly chosen person will have height X between 1.5m and 1.8m? This probability is denoted as $P(1.5m < X < 1.8m)$.

Each CRV has a function $f(x)$, called **probability density (or distribution) function (PDF)**, that describes the relative likelihood for this random variable to take on a given value. PDF has following properties: 1) $f(x) \geq 0, \forall x$ and 2) $\int_{-\infty}^{\infty} f(x) dx = 1$

A probability is given by $P(a \leq X \leq b) = \int_a^b f(x) dx$.

Ex 6. Let
$$f(x) = \begin{cases} 0.006x(10-x) & 0 \leq x \leq 10 \\ 0 & \text{else} \end{cases}, \text{ verify}$$

$$\frac{d}{dx} 0.006x(10-x) = 0.006(10-x) + 0.006x(-1) = 0.006(10-1.006x)$$

it is a probability and find $P(4 \leq X \leq 8)$.

In $[0, 10]$, $f(x)$ is sad parabola with ends at 0, 10, outside of this interval is zero, so it satisfy the first property.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= 0.006 \int_0^{10} 10x - x^2 dx = 0.006 \left(10 \frac{x^2}{2} - \frac{x^3}{3} \right)_0^{10} = 0.006 \left(\frac{10^3}{2} - \frac{10^3}{3} \right) = 1 \\ \int_4^8 f(x) dx &= 0.006 \int_4^8 10x - x^2 dx = 0.006 \left(10 \frac{x^2}{2} - \frac{x^3}{3} \right)_4^8 = 0.006 \left(10 \frac{8^2}{2} - \frac{8^3}{3} - 10 \frac{4^2}{2} + \frac{4^3}{3} \right) = \\ &= 0.006 \left(10 \frac{8^2 - 4^2}{2} + \frac{4^3 - 8^3}{3} \right) \approx 0.544 \end{aligned}$$

The **expectation** of CRV is defined to be a **mean** value of PDF. It can be interpreted as x-coordinate of centroid (center of mass) of the infinite region that lies under PDF. Thus

$$E[X] = \mu = \bar{x} = \frac{\int_{-\infty}^{\infty} xf(x)dx}{\int_{-\infty}^{\infty} f(x)dx} = \int_{-\infty}^{\infty} xf(x)dx$$

Ex 7. Let $f(x) = \begin{cases} 0 & t < 0 \\ Ae^{-ct} & t \geq 0 \end{cases}$. 1) Define A such that f be PDF. 2) Find a mean μ of this PDF. 3) Consider f is a PDF of customer waiting time to a service call. Let $\mu = 5$. What a chance that the customer will be answered during first minute. How many customers (in %) will be answered after the average time of 5 minutes.

1) In order to f be a PDF it should satisfy

$$1 = \int_{-\infty}^{\infty} f(t)dt = \int_0^{\infty} Ae^{-ct} dt = A \lim_{x \rightarrow \infty} \int_0^x e^{-ct} dt = -A \lim_{x \rightarrow \infty} \frac{1}{c} e^{-ct} \Big|_0^x = -\frac{A}{c} \lim_{x \rightarrow \infty} (e^{-cx} - 1) = \frac{A}{c}, \text{ thus } A = c$$

$$2) \quad \mu = \int_{-\infty}^{\infty} tf(t)dt = \lim_{x \rightarrow \infty} \int_0^x tce^{-ct} dt = \lim_{x \rightarrow \infty} \left(-te^{-ct} \Big|_0^x + \int_0^x e^{-ct} dt \right) = \lim_{x \rightarrow \infty} \left(-xe^{-cx} + \frac{e^{-ct}}{-c} \Big|_0^x \right) =$$

$$= \lim_{x \rightarrow \infty} \left(-xe^{-cx} + \frac{e^{-cx}}{-c} - \frac{1}{-c} \right) = \lim_{x \rightarrow \infty} \left(-xe^{-cx} + \frac{1 - e^{-cx}}{c} \right) = \frac{1}{c}$$

$$3) \quad P(0 \leq T \leq 1) = \int_0^1 f(t)dt = \int_0^1 \frac{1}{5} e^{-t/5} dt = \frac{1}{5} (-5) e^{-t/5} \Big|_0^1 = -(e^{-1/5} - 1) = 1 - e^{-1/5} \approx 0.1813$$

$$P(T > 5) = \int_5^{\infty} f(t)dt = \lim_{x \rightarrow \infty} (-e^{-t/5}) \Big|_5^x = -\lim_{x \rightarrow \infty} (e^{-x/5} - e^{-1}) = e^{-1} \approx 0.368 \Rightarrow \approx 37\%$$

Note, that the in recent example, less then half customers will wait more then average time. This is because average is affected from extreme values. In recent example, some customers will wit longer then others, which is extreme waiting time.

Another statistical measure is median, which halves the area under graph of PDF. The

median of PDF is a number m such that $\int_m^{\infty} f(x)dx = \frac{1}{2}$

12.11.6.1 Normal (Gaussian) distribution

A normal distribution is a PDF of the form $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, which is known as Gaussian function or Gaussian bell. The mean value for this function is μ . The positive constant σ is called **standard deviation**; it measures how spread out the values of X are (the smaller the sigma the higher the peak at $x = \mu$).

Note, the Gaussian integral has no elementary antiderivative, therefore it has to be evaluated numerically.