13 Differential Equations

> "… the answer to your question can best be expressed as a series of partial differential equations..."
>
> *Castiel* , Supernatural, Season 6, Episode 18

A differential equation is a mathematical equation for an **unknown function** of one (ODE-Ordinal DE) or several (PDE-Partial DE) variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering, physics, economics, biology, and other disciplines. Many real life processes can be described as differential equations. One of reasons is a way we learn about the process: we investigate a process by observing its change and a rate of the change. In this course we present a quick survey of idea of differential equations in a nutshell.

13.1 Modeling DE (7.1)

13.1.1   Model of population growth

Consider a population problem in ideal condition (unlimited environment and resources, no disease or predator). In such situation we may assume that the population growth is proportional to the population size. Since the rate of population growth is the first derivative of the population size this can be expressed as $\frac{dP}{dt} = kP$, where k is the proportionality constant. If k>0 and initial population is >0 we have $P'(t) > 0$, i.e. the population is always increasing, furthermore the $P'(t)$ increases as the population increases.

The solution to $\frac{dP}{dt} = kP$ is exponential function, i.e. $P(t) = Ce^{kt}$, note that $P'(t) = k(Ce^{kt}) = kP(t)$. The constant C is positive because the nature of the problem (negative population make no sense). We also consider t=0 as initial time and interested in t>0. Note that $P(0) = Ce^{k \cdot 0} = C$, so C is the initial population.

Consider now more complicated model, where the population cannot grow above specific size M (due to limited resources, for example). In this case we look for $P'(t) > 0$ while P<M and $P'(t) < 0$ when P>M. Thus we got $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$. This

equation is known as logistic equation, we will learn to solve it if we get to section 7.5.

Logistic DE have 2 equilibrium solutions P=0 and P=M, from which it never change. When P>M $P'(t)<0$ for 0<P<M $P'(t)>0$ as expected.

13.1.2 Model of the motion of a spring

By Hook's Law, when we release a spring being stretched it exerts force =-kx, from the other side, by Newton's Second Law, the force is mass times acceleration. Thus $m\dfrac{d^2x}{dt^2}=-kx \Rightarrow \dfrac{d^2x}{dt^2}=-\dfrac{k}{m}x$. The second derivative if x is proportional to x with opposite sign, therefore x can be either sin t, cos t or their combination. The presence of trigonometric functions in the solution isn't surprising, since the spring is supposed to oscillate about its equilibrium position.

13.1.3 General DE

In DE we consider y (y(x)) a dependent variable and x independent variable (since the equation should hold for any value of x). In many physical problems independent variable is time, therefore in DE's it often named t, e.g. y'(t) = 3y(t). The order of DE is the highest derivative that occurs in equation, thus it can be formed as following:

First order DE: y'=f(x,y) (read y'(x)=f(x,y(x)).

Second order DE: y''=f(x,y,y')

N'th order DE: $y^{(n)} = f\left(x,y,y',...,y^{(n-1)}\right)$

Examples:

-linear DE $y^{(3)} = ay''(x)+by'(x)+cy(x)+d$

- non linear $y''= y^2(x)$

When we solve DE's we often have a family of solution rather than a single solution, for example

Ex 1. $y' = \cos x$ have infinite number of solutions $\sin x + C$ one per a constant C.

In many physical (or so) problems we often need a particular solution that satisfies an **initial condition** $y(t_0)=y_0$ or a set of IC $y^{(n)}(t_0)=y_{n,0}$. The number of initial condition is equal to order of DE. Such problem is called an **initial value problem**.

Ex 2. Solve $y''=0$ with respect to initial conditions $y(1)=2, y'(1)=1$. From $y''=0$ we learn that y is a line, i.e. $y=ax+b$, applying ICs one get

$y(1)=a\cdot 1+b=a+b=2$ and $y'(1)=b=1\Rightarrow a=1$, thus $y=x+1$ (verify).

Another useful sort of problems called **boundary value problems**, in this case some of conditions are given at the end point of interval of interest t_1. The total number of conditions is still equal to the order of DE.
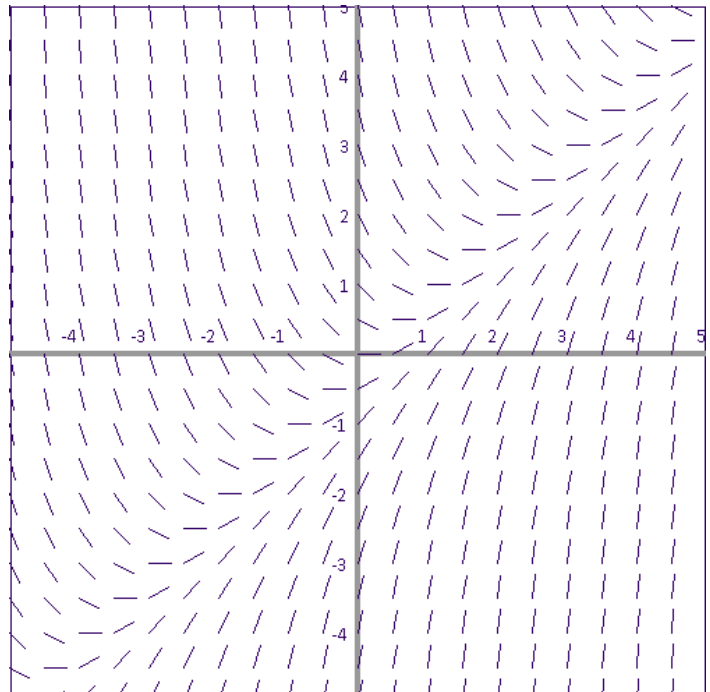
Ex 3. Solve $y''=0$ with respect to initial conditions $y(0)=1, y(1)=-1$. We apply BC to $y=ax+b$ to get $y(0)=a\cdot 0+b=1\Rightarrow b=1$ and $y(1)=a\cdot 1+1=-1\Rightarrow a=-2$, thus $y=-2x+1$ (verify).

13.2 Direction Fields and Euler Method (7.2)

Unfortunately, only certain DE's can be solved in sense of obtaining explicit formula for the solution. Still we can learn a lot about the solution.

13.2.1   Direction Fields

DF is a method to sketch the solution to equation $y'=f(x,y)$, without solve it. Such, graph provide important information about solution's behavior. Note, that we do not look at initial or boundary condition yet, means we looking at the family of solution to DE. The idea is to define a grid and to draw a non crossing little lines or arrows with a slope $m=f(x,y)$ at each grid point. The solution to particular problem $y'=f(x,y)$ with IC $y(x_0)=y_0$ is a thread on a DF graph.

Ex 4. $y'=x-y$
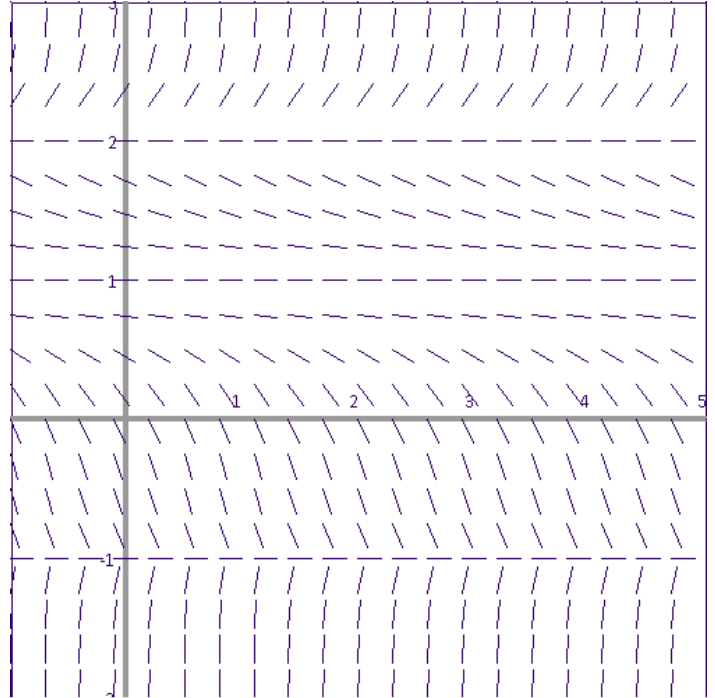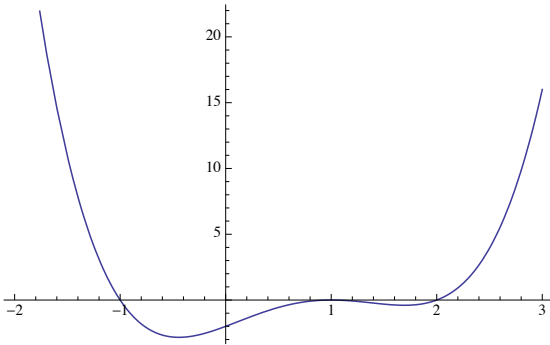
Ex 5.　　$y' = (y^2 - y - 2)(1-y)^2 = (y-2)(y+1)(1-y)^2$

$y < -1 \Rightarrow y' > 0$ (e.g. $y = -2 \Rightarrow y' = 36$)

$-1 < y < 1 \Rightarrow y' < 0$ (e.g. $y = 0 \Rightarrow y' = -2$)

$1 < y < 2 \Rightarrow y' < 0$ (e.g. $y = 1.5 \Rightarrow y' = -0.31$)

$y > 2 \Rightarrow y' > 0$ (e.g. $y = 3 \Rightarrow y' = 16$)

13.2.2　　Euler Method

Consider problem $y' = f(t,y)$ with respect to IC $y(t_0) = y_0$.

Let $h = \Delta t$, denote $t_n = t_0 + nh$, and $y(t_n) = y_n$, then we integrate both sides of the equation to get $\int_{t_k}^{t_{k+1}} y'(t)dt = \int_{t_k}^{t_{k+1}} f(t,y(t))dt$. Evaluating the integral of the left side one get

$y(t_{k+1}) - y(t_k) = \int_{t_k}^{t_{k+1}} f(t,y(t))dt$ or $y_{k+1} = y_k + \int_{t_k}^{t_{k+1}} f(t,y(t))dt$ the last integral one solve using

rectangular rule to get $y_{k+1} = y_k + (t_{k+1} - t_k)f(t_k,y(t_k)) = y_k + hf(t_k,y_k)$ which is called Euler' scheme or method. Different approximations of the integral lead to different schemes, but this is out of the scope of this course.

Ex 6.　　Let solve $\begin{cases} y' = y \\ y(0) = 1 \end{cases}$ (the exact solution is e^x)

The Euler method gives us $y_{n+1} = y_n + hf(x_n, y_n) = y_n + hy_n = y_n(1+h)$

$$y_0 = 1$$
$$y_1 = y_0(1+h)$$
$$y_2 = y_1(1+h) = (1+h)^2$$

Thus $y_n = (1+h)^n$, change $n = \dfrac{x_n}{h}$ to get $y_n = (1+h)^{\frac{x_n}{h}} = \left[(1+h)^{1/h}\right]^{x_n} \underset{h \to 0}{\to} e^{x_n}$

Ex 7.   Use Euler method to approximate $y(1)$ using $h = 1, 1/2$ for $\begin{cases} y' + 2ty = t \\ y(0) = 1 \end{cases}$

$h = 1$: $y_1 = y_0 + ht_0(1 - 2y_0) = y_0 = 1$
$$y_1 = y(x_1) = y(x_0 + h) = y(0+1) = 1$$
$h = 1/2$: $y_2 = y_1 + ht_1(1 - 2y_1) = 1 + h^2(1-2) = 1 - h^2 = 1 - 1/4 = 3/4$
$$y_2 = y(x_2) = y(x_0 + 2h) = y(1) = 3/4 \approx 0.75$$

the analytical solution is $\frac{1}{2}\left(1 + e^{-t^2}\right)$, so $y(1) = \frac{1}{2}(1 + 1/e) \approx 0.684$. One learn from the example that in order to get precise result one need very small h. Further more, when we looking for $y(t_1)$ where $t_1 \gg t_0$ we will need many iterations to get good approximation.