14 Infinite Sequences and Series

14.1 Sequences

Def: A **sequence** (or an **infinite sequence**) is a function $f:\mathbb{N}\to\mathbb{R}$ that often given as $f(n)=a_n$. We will often write sequences as $\{a_n\}_{n=1}^{\infty}=\{a_n\}_{n\in\mathbb{N}}$.

Def: A **subsequence** is a sequence that can be derived from another sequence by deleting some elements without changing the order of the remaining elements.

Ex 1. A constant sequence: $\{a_n\}_{n=1}^{\infty}=\{c\}_{n=1}^{\infty}=c,c,c,c...$

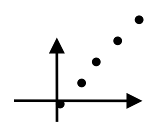
Ex 2. Arithmetic sequence (progression): $\{a_n\}_{n=1}^{\infty}=\{a_0+(n-1)d\}_{n=1}^{\infty}=a_0,a_0+d,a_0+2d...$

Ex 3. Geometric sequence (progression): $\{a_n\}_{n=1}^{\infty}=\{a_1q^{n-1}\}_{n=1}^{\infty}=a_1,a_1q,a_1q^2...$

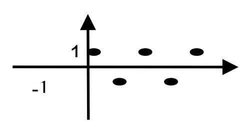
Ex 4. Harmonic sequence: $\{a_n\}_{n=1}^{\infty}=\left\{\frac{1}{n}\right\}_{n=1}^{\infty}=1,\frac{1}{2},\frac{1}{3},\frac{1}{4}...$

Ex 5. Subsequence of Harmonic sequence: $\{a_{2n}\}_{n=1}^{\infty}=\left\{\frac{1}{2n}\right\}_{n=1}^{\infty}=\frac{1}{2},\frac{1}{4},\frac{1}{6}...$

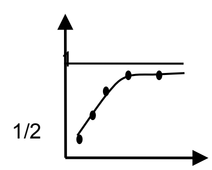
Graphical representation of sequence:



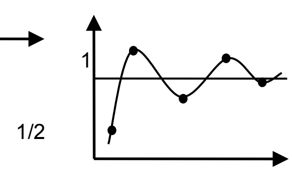
Ex 6. $\{n+1\}=1,2,3,4...$



Ex 7. $\left\{(-1)^{n+1}\right\}=1,-1,1,-1...$



Ex 8. $\left\{\frac{n}{n+1}\right\}=\frac{1}{2},\frac{2}{3},\frac{3}{4}...$



Ex 9. $\left\{1+\left(-\frac{1}{2}\right)^n\right\}=\frac{1}{2},1\frac{1}{4},\frac{7}{8},1\frac{1}{16}$

Ex 10. Write the following sequence as $\{a_n\}_{n=1}^{\infty}$

a. $\frac{1}{2},\frac{3}{4},\frac{5}{6},\frac{7}{8}...=\left\{\frac{2n-1}{2n}\right\}_{n=1}^{\infty}$

b. $\frac{1}{3},\frac{1}{9},\frac{1}{27},\frac{1}{81}...=\left\{\frac{1}{3^n}\right\}_{n=1}^{\infty}$

Def: A sequence $\{a_n\}$ has a limit L, i.e. $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$ if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists we say the sequence converges (convergent). Otherwise the sequence is diverges (divergent).

Ex 11. $\lim_{n\to\infty}(-1)^{n+1} = \text{DNE}$

Ex 12. $\lim_{n\to\infty}\left(1+\left(-\frac{1}{2}\right)^n\right) = 1$

Ex 13.

Thm: If $\lim_{x\to\infty} f(x) = L$, then the sequence $f(n) = a_n$ is convergent and $\lim_{n\to\infty} a_n = L$

Ex 14. $\lim_{n\to\infty} n+1 = \lim_{x\to\infty} x+1 = \infty$

Ex 15. $\lim_{n\to\infty}\frac{n}{n+1} = \lim_{x\to\infty}\frac{x}{x+1} = 1$

Ex 16. $\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{x\to\infty}\frac{\ln x}{x} \underset{L'Hospital\,\infty/\infty}{=} \lim_{x\to\infty}\frac{1/x}{1} = 0$

Limits properties:

If a_n, b_n are convergent and c is constant then

1) $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

2) $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$ including $\lim_{n\to\infty} c = c$

3) $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$

4) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \lim_{n\to\infty} b_n \neq 0$

5) $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p, p > 0, a_n > 0$

The squeeze theorem for sequences: If $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty} b_n = L$

Absolute Value Theorem: $\lim_{n\to\infty}|a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$ (Since $-|a_n| \le a_n \le |a_n|$)

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Thm: If *f* is continuous function at L and $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$

Ex 1. $\lim_{n\to\infty}\left(\frac{\pi}{2}+\left(-\frac{1}{2^n}\right)\right)=\frac{\pi}{2}\Rightarrow \lim_{n\to\infty}\sin\left(\frac{\pi}{2}+\left(-\frac{1}{2^n}\right)\right)=\sin\lim_{n\to\infty}\left(\frac{\pi}{2}+\left(-\frac{1}{2^n}\right)\right)=1$

Ex 2. $\lim_{n\to\infty}\frac{1}{n}=0\Rightarrow f(x)=\frac{1}{x}\Rightarrow f\left(\frac{1}{n}\right)=n\Rightarrow \lim_{n\to\infty} f\left(\frac{1}{n}\right)=\lim_{n\to\infty} n=\infty=\text{"}f(0)\text{"}$

Thm: If a sequence $\{a_n\}_{n=1}^{\infty}$ converges iff subsequences $\{a_{2n}\}_{n=1}^{\infty}$ and $\{a_{2n+1}\}_{n=1}^{\infty}$ does.

Thm: If a sequence $\{a_n\}_{n\in\mathbb{N}}^{\infty}$ converges iff all its subsequences converges.

Corollary: If there exists a divergent subsequence of $\{a_n\}_{n\in\mathbb{N}}^{\infty}$, then $\{a_n\}_{n\in\mathbb{N}}^{\infty}$ diverges.

Thm: $\lim_{n\to\infty} r^n = \begin{cases} 0 & -1<r<1 \\ 1 & r=1 \end{cases}$, When $r>1$ the sequence tends to infinity, and it doesn't exists when $r<-1$ (the last 2 are divergent sequences).

Def: A sequence $\{a_n\}$ is increasing if $a_n \le a_{n+1}$ for all $n\ge 1$. It is called decreasing if it is $a_n \ge a_{n+1}$ for all $n \ge 1$. A sequence is monotonic if it is either increasing or decreasing.

Def: A sequence $\{a_n\}$ is bounded above if there is number M such that $a_n \le M, \forall n \ge 1$. It is bounded below if there is number m such that $a_n \ge m, \forall n\ge 1$. If it is bounded above and below it called bounded sequence.

Thm: Every bounded, monotonic sequence is convergent.

Thm: If $\{b_n\}$ is a subsequence of sequence $\{a_n\}$ obtained by deletion of its first n_0 (finite number) terms. Then $\{a_n\}$ converges iff $\{b_n\}$ does.

Monotonicity tests: 1) $\operatorname{sgn}(a_{n+1}-a_n)$ 2) Does $\frac{a_{n+1}}{a_n}<1$ or $\frac{a_{n+1}}{a_n}>1$?

Ex 3. $n\le n+1\Rightarrow 0\le\frac{n}{n+1}\le 1$. Thus $\left\{\frac{n}{n+1}\right\}$ is bounded and therefore convergent.

Ex 4. Check monotonicity of $(3+5n^2)/(n+n^2)$

$$a_{n+1}-a_n=\frac{3+5(n+1)^2}{(n+1)+(n+1)^2}-\frac{3+5n^2}{n+n^2}=\frac{3+5\left(n^2+2n+1\right)}{(n+1)+\left(n^2+2n+1\right)}-\frac{3+5n^2}{n+n^2}=\frac{5n^2+10n+8}{n^2+3n+2}-\frac{3+5n^2}{n+n^2}=$$

$$=\frac{5n^2+10n+8}{(n+2)(n+1)}-\frac{3+5n^2}{n(n+1)}=\frac{5n^3+10n^2+8n-\left(3+5n^2\right)(n+2)}{n(n+2)(n+1)}=\frac{5n^3+10n^2+8n-3n-5n^3-6-10n^2}{n(n+2)(n+1)}=$$

$$=\frac{5n-6}{n(n+2)(n+1)}>0\Rightarrow 5n-6>0\Rightarrow n>\frac{6}{5}$$

The other test:

$$\frac{a_{n+1}}{a_n}=\frac{3+5(n+1)^2}{(n+1)+(n+1)^2}\frac{n+n^2}{3+5n^2}=\frac{3+5n^2+10n+5}{\cancel{(n+1)}(n+2)}\frac{n\cancel{(1+n)}}{3+5n^2}=\frac{5n^3+10n^2+8n}{5n^3+10n^2+3n+6}>1$$

$$\Leftrightarrow 8n>3n+6\Leftrightarrow 5n>6\Leftrightarrow n>\frac{6}{5}$$

Ex 5. Recursive sequences defined $a_1=10$, $a_{n+1}=(2+a_n)/2$

$$a_1=10, a_2=\frac{2+10}{2}=6, a_3=\frac{2+6}{2}=4....$$

$$L=\lim_{n\to\infty}a_n=\lim_{n\to\infty}a_{n+1}=\lim_{n\to\infty}\tfrac{2+a_n}{2}=1+\tfrac{1}{2}\lim_{n\to\infty}a_n=1+\tfrac{1}{2}L\Rightarrow L=1+\tfrac{1}{2}L\Rightarrow\tfrac{1}{2}L=1\Rightarrow L=2$$

Ex 6. $\lim_{n\to\infty}\left(1+\frac{6}{n}\right)^n=e^6$