PRACTICE Test III Solution

MA141-008 10/31/2018 **Name:** ________________________

Instructor: Michael Medvinsky

TA: Wesley Nelson

Read all of the following information before starting the exam:

- Complete each problem. Show all of your work clearly and in order and justify your answers, as partial credit will be given when appropriate and there may be NO credit given for problems without supporting work. **Circle or otherwise indicate your final answers your final answers.** All answers should be completely simplified, unless otherwise stated.

  You may not use calculator.

1. Use linear approximation to find $f(a+h)$ for $f(x) = \frac{1}{\cos x}$, $a = \frac{\pi}{4}$ and $h = \frac{\pi}{3}$.
Hint: $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$.

Solution: For linear approximation we first need $f'(x) = \frac{\sin x}{\cos^2 x}$, then we use the formula:

$$f(a+h) \approx f(a) + f'(a)h = \frac{1}{\cos a} + h\frac{\sin a}{\cos^2 a} = \sqrt{2} + \frac{\pi}{3}\frac{1/\sqrt{2}}{1/2} = \sqrt{2}(1+\frac{\pi}{3})$$

2. A rectangle with base on the x-axis has its upper vertices on the curve $y = 3 - x^2$. Find the maximum area of such a rectangle.

Solution: The area of the rectangle is given by $A(x) = 2x(3-x^2)$ and it reaches the maximum when $A'(x) = 6 - 6x^2 = 0$. Thus it reaches the maximum at $x = 1$ and the maximal area is 4.

3. Use Newton's method with $x_0 = 1$ to find the root of the equation $x^4 - x^2 = 2$. Show only 2 iterations, i.e. x_1, x_2

Solution: Let $f = x^4 - x^2 - 2$, the Newton iteration is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'x_n} = x_n - \frac{x_n^4 - x_n^2 - 2}{4x_n^3 - 2x_n} = \frac{3x_n^4 - x_n^2 + 2}{4x_n^3 - 2x_n}$$

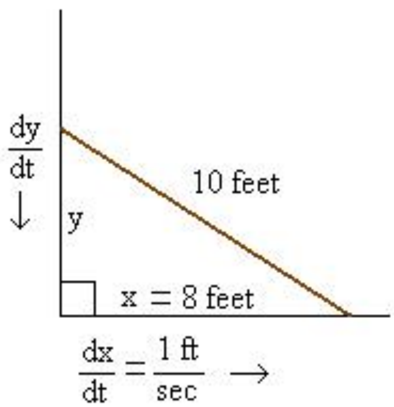
The iterations are:

$$x_0 = 1$$
$$x_1 = \frac{3\cdot 1^4 - 1^2 + 2}{4\cdot 1^3 - 2\cdot 1} = 2$$
$$x_2 = \frac{3\cdot 2^4 - 2^2 + 2}{4\cdot 2^3 - 2\cdot 2} = \frac{23}{14}$$

4. A ladder 10 feet long is resting against a wall. If the bottom of the ladder is sliding away from the wall at a rate of 1 foot per second, how fast is the top of the ladder moving down when the bottom of the ladder is 8 feet from the wall?

Solution:

The picture looks like this.



We have dx/dt = 1 ft/sec, and we want dy/dt. x and y are related by the Pythagorean Theorem

$$x^2 + y^2 = (10\,\text{ft})^2$$

Differentiate both sides of this equation with respect to t to get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When x = 8 ft, we have

$$y = \sqrt{(10\,\text{ft})^2 - (8\,\text{ft})^2} = 6\,\text{ft}$$

therefore

$$\frac{dy}{dt} = -\frac{8\,\text{ft}}{6\,\text{ft}} \cdot \frac{1\,\text{ft}}{\text{sec}} = -\frac{4}{3}\,\frac{\text{ft}}{\text{sec}}$$

The top of the ladder is sliding down (because of the negative sign in the result) at a rate of 4/3 feet per second.

5. Find the following limits or show they do not exist. Show your work in detail.

1. $\lim\limits_{x\to0} \frac{(3x-\sin 3x)^n}{x^{3n}} = \lim\limits_{x\to0} \left(\frac{3x-\sin 3x}{x^3}\right)^n = \left(\lim\limits_{x\to0} \frac{3x-\sin 3x}{x^3}\right)^n \underset{0/0 L'Hospital}{=} \left(\lim\limits_{x\to0} \frac{3-3\cos 3x}{3x^2}\right)^n =$
$\underset{0/0 L'Hospital}{=} \left(\lim\limits_{x\to0} \frac{9\sin 3x}{6x}\right)^n = \frac{9^n}{2^n}\left(\lim\limits_{x\to0} \frac{\sin 3x}{3x}\right)^n = \frac{9^n}{2^n}$

2. $\lim\limits_{x\to0} \frac{1}{x} \ln\sqrt{\frac{1+x}{1-x}} = \lim\limits_{x\to0} \frac{\ln\sqrt{\frac{1+x}{1-x}}}{x} = \frac{1}{2}\lim\limits_{x\to0} \frac{\ln\frac{1+x}{1-x}}{x} = \frac{1}{2}\lim\limits_{x\to0} \frac{\ln(1+x)-\ln(1-x)}{x} =$
$\underset{0/0 L'Hospital}{=} \frac{1}{2}\lim\limits_{x\to0} \frac{\frac{1}{1+x} - \frac{1}{(1-x)}\cdot(-1)}{1} = \frac{1}{2}\cdot(1+1) = 1$

6.

List all critical points, local/global maximum and minimum points, points of inflection, intervals of increasing/decreasing, and intervals of concavity of $f(x) = x^4 - 2x^3$

Solution:

$$f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3) = 0 \Leftrightarrow x = 0 \text{ or } x = \frac{3}{2}$$
$$f''(x) = 12x^2 - 12x = 12x(x-1)$$

The function is increasing when $f'(x) > 0 \Leftrightarrow 2x - 3 > 0 \Leftrightarrow 2x > 3 \Leftrightarrow x > \frac{3}{2}$. The function is decreasing when $x < \frac{3}{2}$. So we have a global minimum at $x = \frac{3}{2}$. The function is concave up if $f''(x) > 0 \Leftrightarrow x(x-1) = x^2 - x > 0$, i.e. when $x < 0$ or $x > 1$. The function is concave down if $f''(x) < 0 \Leftrightarrow 0 \leq x \leq 1$. Thus, there are inflection points at $x = 0$ and $x = 1$.

Good luck!