Final PRACTICE Exam Solution

MA141-008 12/17/2018 **Name:** ________________________
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Read all of the following information before starting the exam:

- Complete each problem. Show all of your work clearly and in order and justify your answers, as partial credit will be given when appropriate and there may be NO credit given for problems without supporting work. **Circle or otherwise indicate your final answers your final answers.** All answers should be completely simplified, unless otherwise stated.

  You may not use calculator.

1. An electrical company at a point A needs to run a wire from a generator to a factory that is on the other side of a one mile wide river and 10 miles downstream at a point C. It costs \$600 per mile to run the wire on towers across the river and \$400 per mile to run the wire over land along the river. The wire will cross the river from A to a point X and then travel over land from X to C. Let x be the distance from a point B directly across the river from point A to the point X. Write a function $C(x)$ representing the total cost in terms of x and use that function to find the value of x that minimizes the cost.

Solution: The cost of the wire is:

$$\text{Cost} = \$600(\text{distance from } A \text{ to } X) + \$400(\text{distance from } X \text{ to } C)$$
$$\boxed{C(x) = 600\sqrt{x^2+1} + 400(10-x)},$$

with domain $[0, 10]$. The cost is a minimum either at a critical point of $f(x)$ or at one of the endpoints of $[0, 10]$. The critical points are solutions to $C'(x) = 0$.

$$\begin{aligned} C'(x) &= 600 \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot (2x) + 400 \cdot (-1) \\ &= \frac{600x}{\sqrt{x^2+1}} - 400 = 0, \end{aligned}$$

which yields $3x = 2\sqrt{x^2+1} \implies x^2 = \frac{4}{5} \implies x = \frac{2}{\sqrt{5}}$. By hand, we can calculate the costs to be the following at each point:

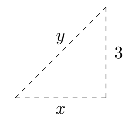
$$\begin{aligned} C\left(2/\sqrt{5}\right) &= 200(20+\sqrt{5}) \\ C(0) &= 4600 \\ C(10) &= 600\sqrt{101} > 600\sqrt{100} = 6000. \end{aligned}$$

Since $200(20+\sqrt{5}) < 200(20+\sqrt{9}) = 200(23) = 4600$, then the minimum cost occurs at $\boxed{x = \frac{2}{\sqrt{5}}}$.

2. A plane is flying directly away from you at 500 mph at an altitude of 3 miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

Solution:

To see what's going on, we first draw a schematic representation of the situation, as in the figure.



Because the plane is in level flight directly away from you, the rate at which x changes is the speed of the plane, $\frac{dx}{dt} = 500$. The distance between you and the plane is y; it is $\frac{dy}{dt}$ that we wish to know. By the Pythagorean Theorem we know that $x^2 + 9 = y^2$. Taking the derivative:

$$2xx' = 2yy'.$$

We are interested in the time at which $x = 4$; at this time we know that $4^2 + 9 = y^2$, so $y = 5$. Putting together all the information we get

$$2(4)(500) = 2(5)y'.$$

Thus, $y' = 400$ mph.

3. Find $\frac{dy}{dx}$ of $y = \arcsin(\arccos x)$

Solution:

$$\sin y = \arccos x$$
$$x = \cos(\sin(y))$$
$$1 = -\sin(\sin(y)) \cdot (\cos(y) \cdot y')$$
$$y' = -\frac{1}{\sin(\sin(y))\cos(y)} = -\frac{1}{\sin(\arccos x)\cos(\arcsin(\arccos x))} =$$
$$= -\frac{1}{\sqrt{1-x^2}\sqrt{1-(\arccos(x))^2}}$$

- The last equality is due to following identity $\cos \arcsin x = \sin \arccos x = \sqrt{1-x^2}$, you do not required to remember it for the quiz or exam, but it is useful, so it worth to remember it.

4. Evaluate following integrals.

1. $\displaystyle\int_0^2 x\sqrt{2x^2+1}dx$

Solution: Substitute $u = 2x^2 + 1$, so that $\frac{1}{4}du = xdx$, to get

$$\int_0^2 x\sqrt{2x^2+1}dx = \frac{1}{4}\int_1^9 \sqrt{u}du = \frac{1}{4}\left(23u^{3/2}\right)_1^9 = \frac{1}{6}(9^{3/2} - 1) = \frac{13}{3}$$

2. $\displaystyle\int \ln(x)x^2dx$

Solution: We solve this integral using integration by parts. Let $u = \ln x$ and $dv = x^2dx$. Then $du = \frac{1}{x}dx$ and $v = \frac{1}{3}x^3$. So we have

$$\int \ln(x)x^2dx = \int udv = uv - \int vdu = \frac{1}{3}x^3\ln x - \int \frac{1}{3}x^3 \cdot \frac{1}{x}dx = \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2dx$$
$$= \boxed{\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C}$$

5. Find the following limits or show that they don't exist.

1. $\lim_{x\to\infty} \frac{\ln^n x}{x} \underset{\infty/\infty L'Hospital}{=} \lim_{x\to\infty} \frac{n\ln^{n-1}x\cdot(1/x)}{1} = \lim_{x\to\infty} \frac{n\ln^{n-1}x}{x} \underset{\infty/\infty L'Hospital}{=} \cdots \underset{\infty/\infty L'Hospital}{=} 1 \cdot 2 \cdots n \cdot \lim_{x\to\infty} \ln x \cdot \frac{1}{x} \underset{\infty/\infty L'Hospital}{=} 1 \cdot 2 \cdots n \cdot \lim_{x\to\infty} \frac{1}{x} = \boxed{0}$

2. $\displaystyle\lim_{x\to 4}\left(\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right)$

Solution:
Upon subsituting $x = 4$ into the function we get the indeterminate form $\infty - \infty$, so let us rewrite the expression in one fraction using the common denominator $x-4 = (\sqrt{x}-2)(\sqrt{x}+2)$ as follows:
$$\left(\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right) = \frac{\sqrt{x}+2-4}{x-4} = \frac{\sqrt{x}-2}{x-4}.$$
We can find the limit as follows:
$$\lim_{x\to 4}\left(\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right) = \lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} \underset{0/0LHR}{=} \lim_{x\to 4}\frac{\frac{1}{2\sqrt{x}}}{1} = \boxed{\frac{1}{4}}$$

3. $\lim_{x\to 0}(\cot x)(x^2+5x) = \lim_{x\to 0}\left(\frac{\cos x}{\sin x}\right)(x^2+5x) = \lim_{x\to 0}\left(\frac{\cos x}{\sin x}\right)x(x+5)$
$= \lim_{x\to 0}\left(\frac{x}{\sin x}\right)(\cos x)(x+5) = \left(\lim_{x\to 0}\frac{x}{\sin x}\right)\left(\lim_{x\to 0}(\cos x)(x+5)\right)$
$= (1)(\cos 0)(0+5) = \boxed{5}$

6. Show that there is a positive real solution to the equation $x^2 + 2 = 10^x$.

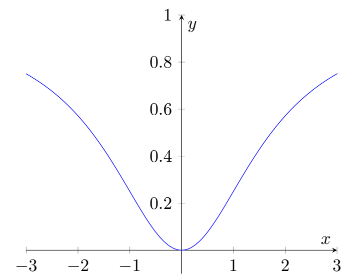
Solution: Let $f(x) = x^2 + 2 - 10^x$. First we recognize that $f(x)$ is continuous everywhere. Next, we must find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs. Let's choose $a = 0$ and $b = 1$. Then we have
$$f(0) = 0^2 + 2 - 10^0 = 1$$
$$f(1) = 1^2 + 2 - 10^1 = -7.$$
Since $f(0) > 0$ and $f(1) < 0$, the Intermediate Value Theorem tells us that $f(c) = 0$ for some c in the interval $(0, 1)$, all of whose elements are positive numbers.

7. Find all critical points, local extrema, inflection points, intervals of increasing and decreasing, and intervals of concavity for the function $f(x) = \dfrac{x^2}{x^2+3}$.

Solution:

First, let us find the critical points (where $f'(x) = 0$ or does not exist). The derivative is

$$f'(x) = \frac{(x^2+3)(2x) - (x^2)(2x)}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}.$$



We can see that the derivative exists everywhere so the only critical point is $x = 0$. Using test points on either side of $x = 0$, we can find that f is increasing on the interval $(0, \infty)$ and decreasing on the interval $(-\infty, 0)$. Also, since f' changes sign from negative to positive at $x = 0$, then this point is a local minimum. To determine concavity and inflection points, let us take the second derivative. We have

$$f''(x) = \frac{(x^2+3)^2 \cdot (6) - (6x)(2)(x^2+3)(2x)}{(x^2+3)^4} = \frac{18(1-x^2)}{(x^2+3)^3}.$$

Again, we can see that the second derivative exists on the entire domain, so the only critical points of f'' are points such that $1 - x^2 = 0$. Thus, we have the critical points $x = -1$ and $x = 1$. Picking test points on the different intervals divided by these critical points, we discover that f is concave up on the interval $(-1, 1)$ and concave down on the set $(-\infty, -1) \cup (1, \infty)$. Since the concavity changes at both points $x = -1$ and $x = 1$, then both are points of concavity. More directly, the points of concavity are $(\pm 1, \frac{1}{4})$.

8. Find the equation of the tangent line to the curve given by $x^2y^2 + y^3 = 2$ at the point $(1, 1)$ on the graph. Express your answer in slope-intercept form, i.e. $y = mx + b$.

Solution: Since we have a point on the line, all we need is to find the slope of the line. Let's solve for the slope by implicitly differentiating the given equation for the curve. We find

$$D_x(x^2y^2 + y^3) = 2xy^2 + x^2 \cdot (2y)\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0 = D_x(2).$$

Rather than solving for $\frac{dy}{dx}$, let us input the values $x = 1$ and $y = 1$ which is a known point on the curve, and then we will solve for the slope. This yields

$$2 + 2y' + 3y' = 0 \implies y' = \frac{-2}{5}.$$

So we have $(y - 1) = -\frac{2}{5}(x - 1)$ which then gives us $\boxed{y = -\frac{2}{5}x + \frac{7}{5}}$ in slope-intercept form.

9. Find the area of the region enclosed by the curves $x + y^2 = 0$ and $x + 3y^2 = 2$.

Solution: Let us first find the points of intersection by setting the x's equal to each other and solving for y, so we have $-y^2 = -3y^2 + 2 \implies y = \pm 1$. So the points of intersection are $(-1, \pm 1)$. From a sketch of these functions we can see that one possibility is to have y be our variable of integration. So we will integrate from $y = -1$ to $y = 1$. From a sketch we can see that the left curve is $x = -y^2$ and the right curve is $x = -3y^2 + 2$ when y is on the interval $(-1, 1)$. So our integral can be written

$$\begin{aligned} A &= \int_{-1}^{1} (\text{right} - \text{left}) dy = \int_{-1}^{1} [(-3y^2 + 2) - (-y^2)] dy \\ &= \int_{-1}^{1} (2 - y^2) dy = \left[2y - \frac{2}{3} y^3 \right]_{-1}^{1} = \boxed{\frac{8}{3}} \end{aligned}$$

10. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region.

1. $y = 1 - x^2$, $y = 0$; about x-axis.

Solution:

$$\pi \int_{-1}^{1} (1 - x^2)^2 dx = \frac{16}{15} \pi$$

2. $y = 7$, $y = \sqrt{x}$, $x = 0$, $x = 4$; about the y axis.

Solution:

$$\pi \int_{0}^{2} (y^2)^2 dy + \pi \int_{2}^{7} 4^2 dy = \frac{432}{5} \pi$$

Good luck!