PRACTICE Test II Solution

MA141-008 09/12/2018 **Name:** ________________________

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Read all of the following information before starting the exam:

- Complete each problem. Show all of your work clearly and in order and justify your answers, as partial credit will be given when appropriate and there may be NO credit given for problems without supporting work. **Circle or otherwise indicate your final answers your final answers.** All answers should be completely simplified, unless otherwise stated.

  You may not use calculator.

1. Find the domain of $\frac{1}{\sin x - 1}$

Solution: The function is defined everywhere but at $x \in A$, where

$$A = \{\sin x = 1\} = \{x = \frac{\pi}{2} + 2\pi n \mid n = \pm 1, \pm 2, \pm 3, \dots\}$$

Therefore the domain is $\mathbb{R} \setminus A$

2. Let $f(x) = \sqrt{2x}$ and $g(x) = x^2 - 1$. Let A be domain of definition of $f \circ g$. Determine if

$$h(x) = \begin{cases} (f \circ g)(x) & x \in A \\ (g \circ f)(x) & \text{else} \end{cases}$$

is continuous in $\mathbb{R}$.

Solution:

$f \circ g = f(g(x)) = f(x^2 - 1) = \sqrt{2(x^2 - 1)}$, the domain is $A = \{|x| \geq 1\}$.

$g \circ f = g(f(x)) = g(\sqrt{2x}) = 2x - 1$

Therefore

$$h(x) = \begin{cases} \sqrt{2(x^2 - 1)} & |x| \geq 1 \\ 2x - 1 & \text{else} \end{cases} = \begin{cases} 2x - 1 & -1 \leq x \leq 1 \\ \sqrt{2(x^2 - 1)} & \text{else} \end{cases}$$

The function $h(x)$ has discontinuity at $x = 1$, since

$$\lim_{x \to 1^+} h(x) = \sqrt{2(1^2 - 1)} = 0 \neq 1 = 2 \cdot 1 - 1 \lim_{x \to 1^-} h(x)$$

Furthermore, the function $h(x)$ has also discontinuity at $x = -1$, since

$$\lim_{x \to -1^-} h(x) = \sqrt{2((-1)^2 - 1)} = 0 \neq -3 = 2 \cdot (-1) - 1 \lim_{x \to -1^+} h(x)$$

Therefore $h(x)$ is a discontinuous function in $\mathbb{R}$.

3. Find the inverse functions to the following functions and state the domain D and range R of both:

   (a) $f(x) = \log_{10}(x^2 + 10)$

   Solution: $f : D = \mathbb{R}, \quad R = \{y | y \geq 1\}$
   Note, that $\log_{10}((3\sqrt{(10)})^2 + 10) = 2 = \log_{10}((-3\sqrt{(10)})^2 + 10)$, therefore the function isn't one-to-one, which means it is non invertible.

   (b) $f(x) = \frac{1}{1+e^{-x}}$

   Solution:

   $$f : D = \mathbb{R}, \quad R = \{0 < y < 1\},$$
   $$f^{-1}(x) = \ln\left(\frac{x}{1-x}\right)$$
   $$f^{-1} : D = \{0 < x < 1\}, \quad R = \mathbb{R}$$

4. Find the following limits or show they do not exist. Show your work in detail.

   (a) $\lim_{x\to 0} \sqrt{\frac{9}{x^2} + \frac{1}{x}} - \frac{3}{x} = \lim_{x\to 0} \frac{\frac{9}{x^2}+\frac{1}{x}-\frac{9}{x^2}}{\sqrt{\frac{9}{x^2}+\frac{1}{x}}+\frac{3}{x}} = \lim_{x\to 0} \frac{1}{x\sqrt{\frac{9}{x^2}+\frac{1}{x}}+3} = \lim_{x\to 0} \frac{1}{\sqrt{9+x}+3} = \frac{1}{6}$

   (b) $\lim_{x\to 0} \frac{\sqrt{1-x^2}-1}{x^2} = \lim_{x\to 0} \frac{1-x^2-1}{x^2(\sqrt{1-x^2}+1)} = \lim_{x\to 0} \frac{-1}{\sqrt{1-x^2}+1} = -\frac{1}{2}$

   (c) $\lim_{x\to\infty} (1 - \frac{1}{2x})^{3x} = \lim_{x\to\infty} \left((1 + \frac{1}{-2x})^{-2x}\right)^{-3/2} = \left(\lim_{x\to\infty} (1 + \frac{1}{-2x})^{\frac{1}{-2x}}\right)^{-3/2} = e^{-3/2}$

   (d) $\lim_{x\to 0} x^2 e^{\sin\frac{1}{x}}$

   Solution: Since $-1 \leq \sin y \leq 1$ for any $y \in \mathbb{R}$ this also true for $y = \frac{1}{x} \in \mathbb{R}$. Conclude that $e^{-1} \leq e^{\sin\frac{1}{x}} \leq e$ and therefore

   $$0 \leftarrow x^2 e^{-1} \leq x^2 e^{\sin\frac{1}{x}} \leq ex^2 \to 0$$

   Thus, by squeeze theorem
   $$\lim_{x\to 0} x^2 e^{\sin\frac{1}{x}} = 0$$

   (e) $\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{1-\lim_{x\to\infty}\frac{1}{x^2}}{1+\lim_{x\to\infty}\frac{1}{x^2}} = \frac{1-0}{1+0} = 1$

5. Find first derivative ($\frac{dy}{dx}$) of the following functions

   (a) $y = \left(\frac{1}{x} + x\right)\left(\frac{1}{x} - x^2\right)$

Solution:

$$\begin{aligned}\frac{dy}{dx} &= \left(\frac{1}{x}+x\right)'\left(\frac{1}{x}-x^2\right)+\left(\frac{1}{x}+x\right)\left(\frac{1}{x}-x^2\right)' \\ &= \left(-\frac{1}{x^2}+1\right)\left(\frac{1}{x}-x^2\right)+\left(\frac{1}{x}+x\right)\left(-\frac{1}{x^2}-2x\right) \\ &= -2-\frac{1}{x^3}-\frac{1}{x}-2x^2+1-\frac{1}{x^3}+\frac{1}{x}-x^2 = -\frac{2+x^3+3x^5}{x^3}\end{aligned}$$

(b) $y = \left[\left(\sqrt{x^2+1}+\frac{x}{\sqrt[3]{x}+1}\right)^{7/3}\right] =$

Solution:

$$= \frac{7}{3}\left(\sqrt{x^2+1}+\frac{x}{\sqrt[3]{x}+1}\right)^{4/3}\cdot\frac{d}{dx}\left(\sqrt{x^2+1}+\frac{x}{\sqrt[3]{x}+1}\right) =$$

$$= \frac{7}{3}\left(\sqrt{x^2+1}+\frac{x}{\sqrt[3]{x}+1}\right)^{4/3}\cdot\left(\frac{2x}{2\sqrt{x^2+1}}+\frac{\sqrt[3]{x}+1-x(\frac{1}{3}x^{-2/3})}{(\sqrt[3]{x}+1)^2}\right)$$

$$= \frac{7}{3}\left(\sqrt{x^2+1}+\frac{x}{\sqrt[3]{x}+1}\right)^{4/3}\cdot\left(\frac{x}{\sqrt{x^2+1}}+\frac{\sqrt[3]{x}+1-\frac{1}{3}\sqrt[3]{x}}{(\sqrt[3]{x}+1)^2}\right)$$

$$= \frac{7}{3}\left(\sqrt{x^2+1}+\frac{x}{\sqrt[3]{x}+1}\right)^{4/3}\cdot\left(\frac{x}{\sqrt{x^2+1}}+\frac{3+2\sqrt[3]{x}}{3(\sqrt[3]{x}+1)^2}\right)$$

(c) $y = \arctan(1+\cos x)$

Solution:

$$\begin{aligned}\tan y &= 1+\cos x \\ \frac{y'}{\cos^2 y} &= -\sin x \\ y' = -\sin x\cos^2 y &= -\sin x\cos^2(\arctan(1+\cos x))\end{aligned}$$

Since $\cos^2\arctan x = \frac{1}{1+x^2}$ we get

$$y' = -\frac{\sin x}{1+(1+\cos x)^2}$$

6. Find an equation of the tangent line to the curve given by

$$x^2+xy+2y^2 = 4$$

at point (1,1).

Solution:

We first need to find $\frac{dy}{dx}$ using implicit differentiation: $2x + y + xy' + 4yy' = 0$, and hence $y' = -\frac{2x+y}{x+4y}$. At the point (1,1), the value of $\frac{dy}{dx}$ is $y'(1) = -\frac{2(1)+(1)}{(1)+4(1)} = \frac{-3}{5}$. The slope of the tangent line at (1,1) is $m = \frac{dy}{dx}\Big|_{(1,1)}$. Finally the tangent line is given by $y - 1 = -\frac{3}{5}(x - 1)$ or $y = -\frac{3}{5}(x - 1) + 1 = -\frac{3}{5}x + \frac{8}{5}$.

Good luck!