Course: MA241, Calculus II	Instructor Michael Medvinsky

Handout 4

Completion the square

$$x^2+bx+c=\left(x+\frac{b}{2}\right)^2+\left(c-\left(\frac{b}{2}\right)^2\right)$$

Or in a more general case, given $a\neq 0$

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left(\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\left(\frac{b}{2a}\right)^2\right)\right)=a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)$$

Partial Fraction

Consider $f(x)=\frac{P(x)}{Q(x)}$ where $P(x),Q(x)$ polynomials. One rewrite $f(x)$ as a sum of simple fractions as following.

- If $Q(x)$ has a factor $(x-r)$ add $\frac{A}{x-r}$.
- If $Q(x)$ has a factor $(x-r)^k$ add $\frac{A_1}{x-r}+\frac{A_2}{(x-r)^2}+\cdots+\frac{A_k}{(x-r)^k}$
- If $Q(x)$ has a factor (ax^2+bx+c) with $b^2-4ac<0$ then add $\frac{Ax+B}{ax^2+bx+c}$
- Similarly for $(ax^2+bx+c)^k$ add $\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

To find the coefficients in the nominator one combine the new form using common denominator, then match coefficients of x^k's with original nominator as in the following example.

Example: $\frac{2x+3}{x^2+x-6}=\frac{2x+3}{(x+3)(x-2)}=\frac{a}{x+3}+\frac{b}{x-2}=\frac{a(x-2)+b(x+3)}{(x+3)(x-2)}=\frac{(a+b)x+(3b-2a)}{(x+3)(x-2)}$, thus

$2x+3=(a+b)x+(3b-2a)$ or $\begin{cases}a+b=2\\3b-2a=3\end{cases}$ which has a solution $b=7/5$ and $a=3/5$

Thus, in order to integrate $f(x)=\frac{P(x)}{Q(x)}$ where $P(x),Q(x)$ polynomials.one need to integrate the following

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1) $\int \frac{A}{x-a}dx = A\ln(x-a)+c$

2) $\int \frac{A}{(x-a)^n}dx = A\cdot\frac{1}{1-n}(x-a)^{1-n}+c, n>1$

3) $\int \frac{Ax+B}{x^2+px+q}dx = \frac{2B-Ap}{\sqrt{4q-p^2}}\cdot arctan\left(\frac{2x+p}{\sqrt{4q-p^2}}\right)$

4) $\int \frac{Ax+B}{\left(x^2+px+q\right)^k}dx$

In order to integrate the last integral one notice that $\frac{d}{dx}\left(x^2+px+q\right)=2x+p$ and rewrite the nominator to match it, thus

$$Ax+B = A\left(x+\frac{B}{A}\right)=\frac{A}{2}\left(2x+2\frac{B}{A}\right)=\frac{A}{2}\left(2x+2\frac{B}{A}+p-p\right)=\frac{A}{2}(2x+p)+\frac{A}{2}\left(2\frac{B}{A}-p\right)=\frac{A}{2}(2x+p)+\left(B-\frac{Ap}{2}\right)$$

In addition one complete the square in the denominator $x^2+px+q=\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)$, thus

$$\int \frac{Ax+B}{\left(x^2+px+q\right)^k}dx = \int \frac{\frac{A}{2}(2x+p)+\left(B-\frac{Ap}{2}\right)}{\left(\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right)^k}dx =$$

$$=\frac{A}{2}\int \frac{(2x+p)}{\left(\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right)^k}dx+\left(B-\frac{Ap}{2}\right)\int \frac{1}{\left(\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right)^k}dx$$

$$\underset{\substack{t=x^2+px+q\\ dt=(2x+p)dx\\ u=x+p/2\\ du=dx\\ \alpha=\sqrt{q-\frac{p^2}{4}}}}{=} \frac{A}{2}\int\frac{dt}{t^k}+\left(B-\frac{Ap}{2}\right)\int\frac{du}{\left(u^2+\alpha^2\right)^k}$$

Thus one also need to integrate $I_k=\int\frac{du}{\left(u^2+\alpha^2\right)^k}$ which is can be found using recurrent technique to get

$$I_{k+1}=\frac{1}{2k\alpha^2}\left(\frac{u}{\left(u^2+\alpha^2\right)^2}+(2k-1)I_k\right)$$, where

$$I_1=\int\frac{du}{u^2+\alpha^2}=\frac{1}{\alpha^2}\int\frac{du}{\left(\frac{u}{\alpha}\right)^2+1}=\frac{1}{\alpha}\int\frac{dt}{t^2+1}=\frac{1}{\alpha}\arctan t=\frac{1}{\alpha}\arctan\frac{u}{\alpha}$$