Midterm II Exam Solution

MA241-011
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Instructor Medvinsky Michael

Name: ____________________________

Read all of the following information before starting the exam:

- Complete each problem. Show all of your work clearly and in order and justify your answers, as partial credit will be given when appropriate and there may be NO credit given for problems without supporting work.
- **Circle or otherwise indicate your final answers..** All answers should be completely simplified, unless otherwise stated.
- **You may use a *non graphical* calculator.**
- This test has 3 problems - you have to solve them all.
- It is your responsibility to make sure that you have all of the pages!
- Good luck!

Problem	Score
1	
2	
3	
TOTAL	

1. Integrate following integrals using partial fractions:

   (a) $\displaystyle\int \frac{1}{4x^3 - x} dx =$

Solution: Start with

$$\begin{aligned}\frac{1}{4x^3-x} =& \frac{1}{x(2x-1)(2x+1)} = \frac{a}{x} + \frac{b}{2x-1} + \frac{c}{2x+1}\\ =& \frac{a(4x^2-1)+bx(2x+1)+cx(2x-1)}{x(2x-1)(2x+1)} = \frac{(4a+2b+2c)x^2+(b-c)x-a}{x(2x-1)(2x+1)}\end{aligned}$$

to get linear system of equations

$$\begin{array}{ccccccc} 4a & + & 2b & + & 2c & = & 0\\ & & b & - & c & = & 0\\ a & & & & & = & 1 \end{array}$$

which has a solution $a = -1$, $b = 1 = c$. Thus,

$$\begin{aligned}\int \frac{dx}{4x^3-x} =& -\int \frac{dx}{x} + \int \frac{dx}{2x-1} + \int \frac{dx}{2x+1}\\ =& -\ln|x| + \frac{1}{2}\ln|2x-1| + \frac{1}{2}|\ln|2x+1| = \frac{1}{2}\ln\left|\frac{4x^2-1}{x}\right|\end{aligned}$$

   (b) $\displaystyle\int \frac{dx}{(x+1)(x^2-x+1)} =$

Solution: Start with

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)}$$

Thus, $1 = A(x^2-x+1)+(Bx+C)(x+1) = (A+B)x^2+(-A+B+C)x+(A+C)$ and we get the following linear system of equations

$$\begin{array}{ccccccc} A & + & B & & & = & 0\\ -A & + & B & + & C & = & 0\\ A & & & + & C & = & 0 \end{array}$$

which has a solution $A = \frac{1}{3}$, $B = -\frac{1}{3}$ and $\frac{2}{3}$. Therefore

$$\frac{1}{3}\cdot\frac{-x+2}{x^2-x+1} = \frac{1}{6}\cdot\frac{-2x+4}{x^2-x+1} = \frac{1}{6}\cdot\frac{-2x+1+3}{x^2-x+1} = -\frac{1}{6}\cdot\frac{2x-1}{x^2-x+1} + \frac{1}{2}\cdot\frac{1}{x^2-x+1}$$

Finally,

$$\begin{aligned}\int \frac{dx}{(x+1)(x^2-x+1)} =& \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{6}\int \frac{2x-1}{x^2-x+1} + \frac{1}{6}\int \frac{1}{x^2-x+1}\\ =& \frac{1}{3}\ln|x+1| - \ln\underbrace{(x^2-x+1)}_{>0} + \frac{1}{6}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\\ =& \frac{1}{3}\ln|x+1|\ln(x^2-x+1) + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C\end{aligned}$$

2. Compute following improper integrals:

(a) $\displaystyle\int\limits_0^{+\infty} (1-x)e^{-x}dx =$

Solution:

$$\int\limits_0^{+\infty} (1-x)e^{-x}dx = \lim_{w\to\infty} \int\limits_0^{w} (1-x)e^{-x}dx = \lim_{w\to\infty}\left((x-1)e^{-x}|_0^w - \int\limits_0^w e^{-x}dx \right)$$

$$= \lim_{w\to\infty} \left((w-1)e^{-w} - (0-1)e^{-0} + \left(e^{-x}\right)|_0^w\right)$$

$$= \lim_{w\to\infty} \left(we^{-w} - e^{-w} + 1 + e^{-w} - 1\right)$$

$$= \lim_{w\to\infty} \left(we^{-w}\right) = \lim_{w\to\infty}\left(\frac{w}{e^w}\right) \underset{L}{=} \lim_{w\to\infty}\left(\frac{1}{e^w}\right) = 0$$

(b) $\displaystyle\int_1^5 \frac{dx}{x^2\sqrt{25-x^2}}$

Solution:

$$\int_1^5 \frac{dx}{x^2\sqrt{25-x^2}} = \int_1^5 \frac{dx}{x^2\cdot 5\sqrt{1-\left(\frac{x}{5}\right)^2}} = \frac{1}{125}\int_1^5 \frac{dx}{\left(\frac{x}{5}\right)^2\sqrt{1-\left(\frac{x}{5}\right)^2}}$$

$$\underset{\substack{t=x/5\\ dt=dx/5}}{=} \frac{1}{25}\int_{1/5}^1 \frac{dt}{t^2\sqrt{1-t^2}} \underset{\substack{t=\sin u\\ dt=\cos u du}}{=} \frac{1}{25}\int_{\arcsin\frac{1}{5}}^{\pi} \frac{\cos u du}{\sin^2 u\sqrt{1-\sin^2 u}}$$

$$= \frac{1}{25}\int_{\arcsin\frac{1}{5}}^{\pi} \frac{du}{\sin^2 u} = -\tan^{-1} u\ \Big|_{\arcsin\frac{1}{5}}^{\pi}$$

3. Use trigonometric substitution

Hint: If $t = \tan\frac{x}{2}$ then $dx = \frac{2dt}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$,

If $t = \tan x$, then $dt = \frac{dx}{\cos^2 x} = (1+\tan^2 x)dx = (1+t^2)dx$, $\cos x = \frac{1}{\sqrt{1+t^2}}$ and $\sin x = \frac{t}{\sqrt{1+t^2}}$

(a) $\displaystyle\int \frac{dx}{3-\cos x+\sin x} =$

Solution:

$$\begin{aligned}\int \frac{dx}{3-\cos x+\sin x} &= \int \frac{dy}{3\,(1+y^2)-(1-y^2)+2y} = \int \frac{dy}{4y^2+2y+2}\\ &= \frac{1}{4}\int \frac{dy}{\left(y+\frac{1}{4}\right)^2+\frac{7}{16}} = \frac{1}{\sqrt{7}}\arctan\left(\frac{4}{\sqrt{7}}\left(y+\frac{1}{4}\right)\right)+C\end{aligned}$$

(b) $\displaystyle\int \sin^4 x\cos^4 x =$

Solution:

$$\int \sin^4 x\cos^4 x = \frac{1}{4}\int \sin^4 2x = \frac{1}{16}\int (1-\cos 4x)^2 = \frac{1}{16}\int 1-2\cos 4x+\cos^2 4x =$$

$$= \frac{x}{16}-\frac{\sin 4x}{32}+\frac{1}{32}\int 1+\cos 8x = \frac{3x}{32}-\frac{\sin 4x}{32}+\frac{\sin 8x}{128}+C$$

(c) $\displaystyle\int \frac{\sin x\cos x}{1+\sin^2 x}dx =$

Solution:

$$\int \frac{\sin x\cos x}{1+\sin^2 x}dx \underset{t=\sin^2 x}{=} \int \frac{dt/2}{1+t} = \frac{1}{2}\ln|1+t|$$

extra space