Double Integral

in general domain (cont)

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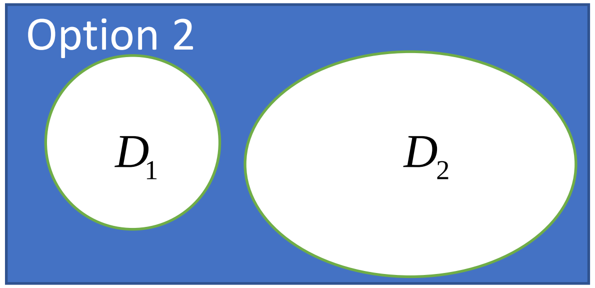
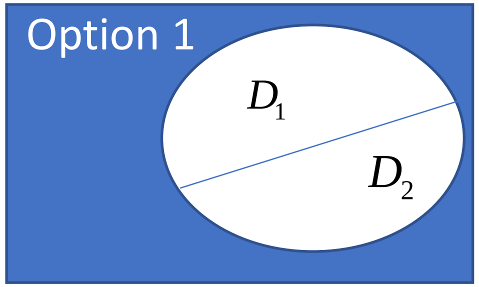
Properties of Double Integral

- Similar to Rectangular Domain
  - $\iint_D f(x,y) + g(x,y) dA = \iint_D f(x,y) dA + \iint_D g(x,y) dA$
  - $\iint_D cf(x,y) dA = c \iint_D f(x,y) dA$
  - $f(x,y) \geq g(x,y) \Rightarrow \iint_D f(x,y) dA \geq \iint_D g(x,y) dA$

Properties of Double Integral

- New
  - $D = D_1 \cup D_2 \Rightarrow \iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$
    - analogy to 1D $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
  - $\iint_D dA = A(D), \textit{i.e. the area of the domain } D$
    - analogy to 1D $\int_a^b dx = b - a$
  - $m \le f(x,y) \le M \Rightarrow mA(D) \le \iint_D f(x,y)dA \le MA(D)$





Properties of Double Integral(example)

- Let D be disc of radius 2. Estimate $\iint\limits_D \sin xy\, dA$

$$-1 \le \sin xy \le 1 \text{ and } A(D) = \pi r^2 = 4\pi \Rightarrow -4\pi \le \iint\limits_D \sin xy\, dA \le 4\pi$$

Double Integral in Polar Coordinates

- Recall:
  - Change of variables in single integral

$$\int_a^b f\left(g(x)\right)g'(x)\,dx \underset{\substack{t=g(x)\\ dt=g'(x)dx}}{=} \int_{g(a)}^{g(b)} f(t)\,dt$$

  - differential

$$df(x,y) = f_x(x,y)\,dx + f_y(x,y)\,dy$$

- Consider differential as a vector base (like the $\hat{\imath},\hat{\jmath},\hat{k}$) and rewrite

$$\overrightarrow{df}(x,y) = f_x(x,y)\,\overrightarrow{dx} + f_y(x,y)\,\overrightarrow{dy}$$

Double Integral in Polar Coordinates(cont)

- The new formulation gives us

$$x = r\cos\theta$$

$$\Rightarrow \overrightarrow{dx} = \cos\theta\overrightarrow{dr} - r\sin\theta\overrightarrow{d\theta} = \langle \cos\theta, -r\sin\theta \rangle$$

- And

$$y = r\sin\theta$$

$$\Rightarrow \overrightarrow{dy} = \sin\theta\overrightarrow{dr} + r\cos\theta\overrightarrow{d\theta} = \langle \sin\theta, r\cos\theta \rangle$$

- Hence the area element becomes:
  - in polar coordinates:

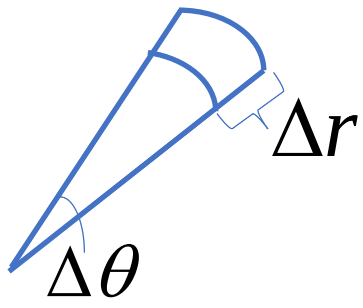
$$A = \left|\overrightarrow{dx} \times \overrightarrow{dy}\right| = \left|\langle \cos\theta, -r\sin\theta, 0 \rangle \times \langle \sin\theta, r\cos\theta, 0 \rangle\right| = \left|\langle 0, 0, r\cos^2\theta + r\sin^2\theta \rangle\right| = r$$

  - in cartesian coordinates is still works

$$A = \left|\overrightarrow{dx} \times \overrightarrow{dy}\right| = |i \times j| = |k| = 1$$

Double Integral in Polar Coordinates(cont)

- **Conclude** $dxdy = rdrd\theta$



- Another way to get this result: Find the area of polar "rectangle" $[r_{i-1}, r_i]\times[\theta_{j-1}, \theta_j]$

$$\begin{aligned}
\Delta A_i &= \frac{\Delta\theta}{2\pi}\pi r_i^2 - \frac{\Delta\theta}{2\pi}\pi r_{i-1}^2 = \frac{\Delta\theta}{2}\left(r_i^2 - r_{i-1}^2\right) \\
&= \frac{\Delta\theta}{2}\left(r_i - r_{i-1}\right)\left(r_i + r_{i-1}\right) = \frac{\left(r_i + r_{i-1}\right)}{2}\left(r_i - r_{i-1}\right)\Delta\theta \\
&= \overline{r}_i \Delta r \Delta\theta
\end{aligned}$$

Double Integral in Polar Coordinates(cont)

- **Theorem**: Let *f(x,y)* be continuous function on a polar rectangle *R* given by $0 \le a \le r \le b, \alpha \le \theta \le \beta,\ 0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

- **Theorem**: Let *f(x,y)* be continuous function on a polar region $D = \{(r,\theta) : \alpha \le \theta \le \beta, 0 \le \beta - \alpha \le 2\pi, h_1(\theta) \le r \le h_2(\theta)\}$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Double Integral in Polar Coordinates(example)

- 1)

$$\iint_{\left\{\begin{matrix} x<0 \\ 3\le\sqrt{x^2+y^2}\le 5 \end{matrix}\right\}} 3x+2ydA = \int_{\pi/2}^{3\pi/2}\int_{3}^{5}\left(3r\cos\theta+2r\sin\theta\right)rdrd\theta$$

$$= \int_{\pi/2}^{3\pi/2}\frac{r^3}{3}\left(3\cos\theta+2\sin\theta\right)\bigg|_3^5 d\theta = \int_{\pi/2}^{3\pi/2}\frac{5^3-3^3}{3}\left(3\cos\theta+2\sin\theta\right)d\theta$$

$$= \frac{98}{3}\left(3\sin\theta-2\cos\theta\right)\bigg|_{\pi/2}^{3\pi/2} = \frac{98}{3}\left\{3\left(-1-1\right)\right\} - \cdot 0 = -196$$

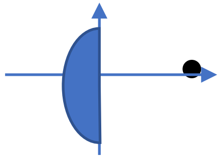
Double Integral in Polar Coordinates(example)

- 2)

$$\iint_{\left\{\begin{smallmatrix}0\le\theta\le\pi\\0\le r\le\sin\theta\end{smallmatrix}\right\}} 3\sqrt{x^2+y^2}-2dA=\int_0^{\pi}\int_0^{\sin\theta}(3r-2)rdrd\theta=\int_0^{\pi} r^3-r^2\Big|_0^{\sin\theta}d\theta$$

$$=\int_0^{\pi}\sin^3\theta-\sin^2\theta d\theta$$

$$=-\frac{1}{3}\left(2+\sin^2\theta\right)\cos\theta-\frac{1}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi}=\frac{4}{3}-\frac{\pi}{2}$$



- 3)

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{0} e^{1-x^2-y^2}dxdy=\int_{\pi/2}^{3\pi/2}\int_0^2 e^{1-r^2}rdrd\theta=\int_{\pi/2}^{3\pi/2}\left(-\frac{1}{2}e^{1-r^2}\right)_0^2 d\theta=-\frac{\pi}{2}\left(e^{-3}-e\right)$$