# Vector Calculus

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# Vector Field (definition)

- **Definition**: Vector Field is a function F that for each (x,y)\(x,y,z) assign a 2\3-dimensional vector, respectively:

$$\mathbf{F}(x,y)=\langle P(x,y),Q(x,y)\rangle$$

$$\mathbf{F}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$$

- Examples of VF: gradient, direction field of differential equation.
- Vector field vs other functions we learned:

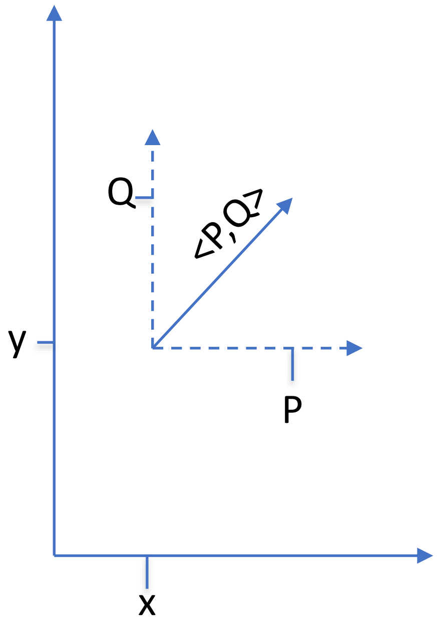
| | |
|---|---|
| $f:\mathbb{R}^n\to\mathbb{R}$ | function of $n=1,2,3$ variables |
| $\vec{r}:\mathbb{R}\to\mathbb{R}^n$ | vector (of size $n=1,2,3$) valued function, e.g. parametric curve |
| $\vec{r}:\mathbb{R}^2\to\mathbb{R}^3$ | parametric surface |
| $\mathbf{F}:\mathbb{R}^n\to\mathbb{R}^n$ | vector field ($n=2,3$) |

# Vector Field (how to sketch it)

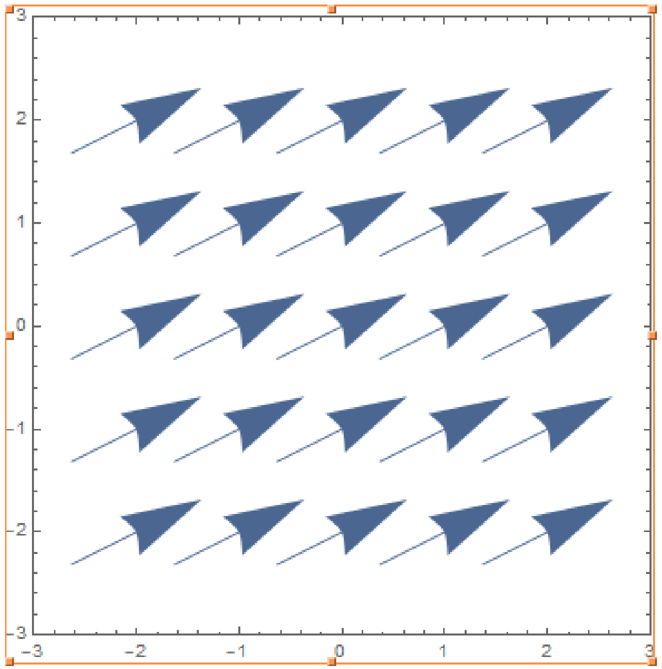
- We draw VF as vectors $\langle P(x,y),Q(x,y)\rangle \backslash \langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$ starting at points $(x,y)\backslash(x,y,z)$

Examples:

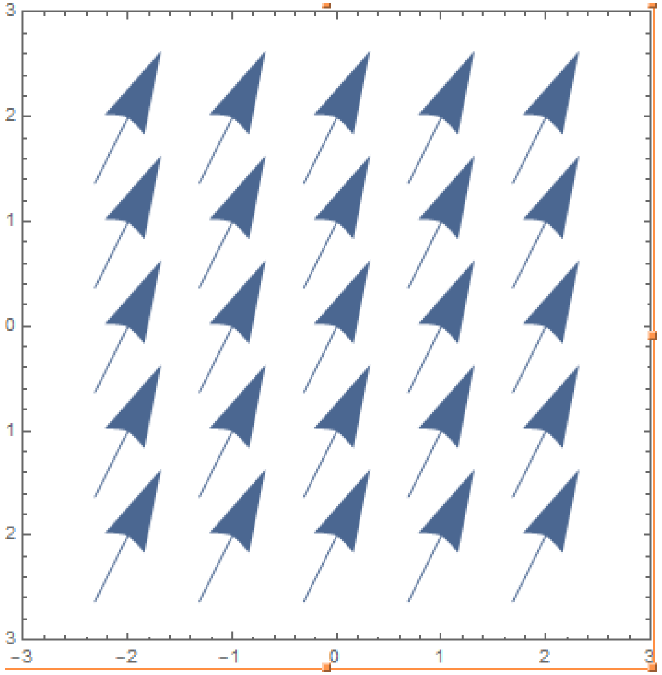
General case




Uniform\Constant VF: F=<2,1>



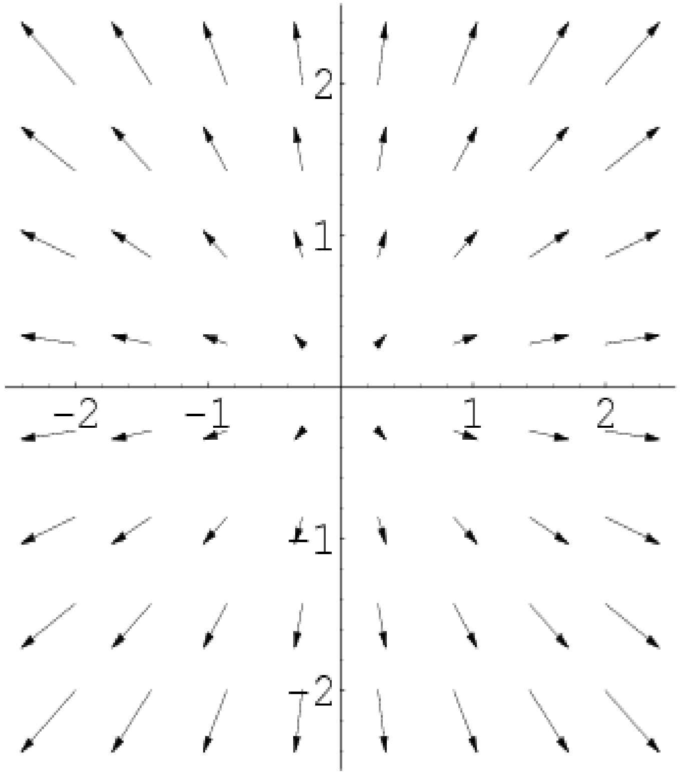

Uniform\Constant VF:F=<1,2>



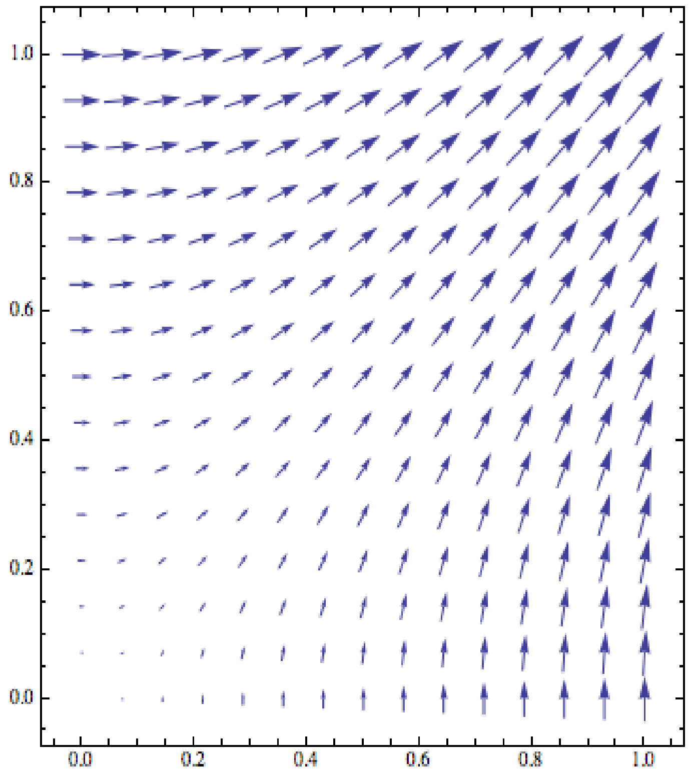
# Vector Field (how to sketch it)

- More examples:

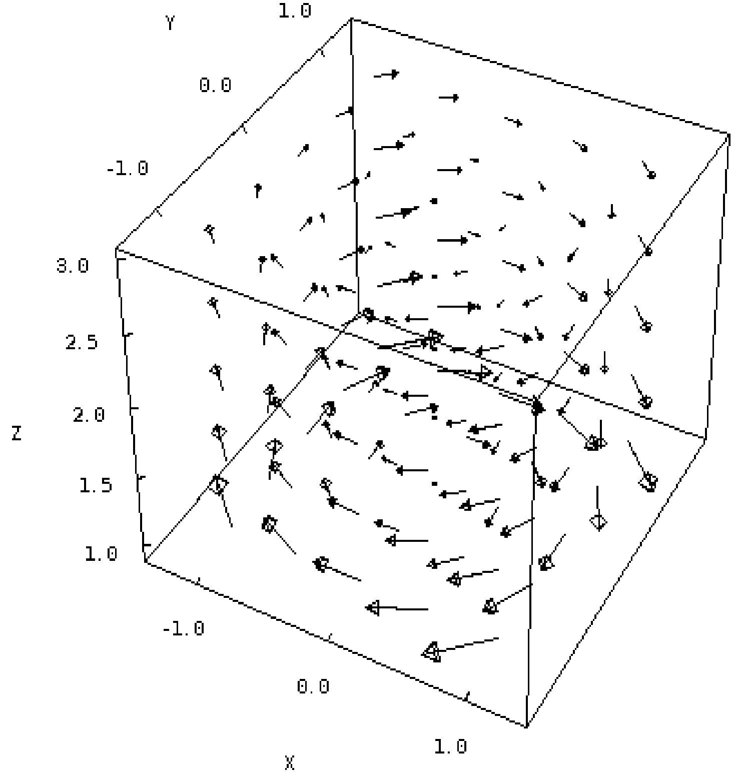
$$\mathbf{F}(x, y) = \langle x, y \rangle$$




$$f(x, y) = xy, \mathbf{F} = \vec{\nabla} f = \langle y, x \rangle$$

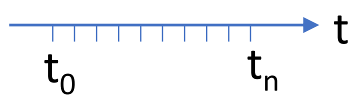



$$\mathbf{F}(x, y, z) = \left\langle \frac{y}{z}, -\frac{x}{z}, 0 \right\rangle$$



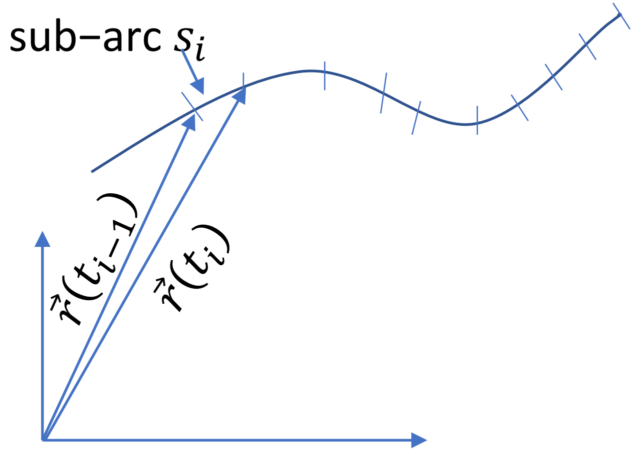
# Line Integral (the idea)

- Consider smooth curve *C* be given by: $\vec{r}(t) = \langle x(t), y(t) \rangle, t \in [a,b]$
  **recall**: smooth means $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t) \neq 0$
- Divide [a,b] into subintervals $t_i = a + i\Delta t, \Delta t = \dfrac{b-a}{n}$




- Denote $s_i$ a piece of *C* corresponding to $[t_{i-1}, t_i]$ and displacement as $\Delta s_i$ .
- Denote $\langle x_i^*, y_i^* \rangle$ a sample point on $s_i$.
- Consider that function *f(x,y)* is defined along *C.*
- What do you think about the following *Rieman-like* sum

$$\sum_{i=1}^{n} f\left(x_i^*, y_i^*\right) \Delta s_i$$



# Line Integral (definition)

- Let *f* be a function defined along a curve C given by

$$\vec{r}\left(t\right)=\left\langle x\left(t\right),y\left(t\right)\right\rangle \quad or \quad \vec{r}\left(t\right)=\left\langle x\left(t\right),y\left(t\right),z\left(t\right)\right\rangle, \quad t\in\left[a,b\right]$$

then the line\contour\path\curve integral is defined by

$$\oint_C f\left(x,y\right)ds=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*,y_i^*\right)\Delta s_i \quad or \quad \oint_C f\left(x,y,z\right)ds=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*,y_i^*,z_i^*\right)\Delta s_i$$

- *Recall: Arclength formula:*

$$L=\int_a^b\sqrt{x_t^2+y_t^2}\,dt \quad or \quad L=\int_a^b\sqrt{x_t^2+y_t^2+z_t^2}\,dt \quad or \quad L=\int_a^b\left|\vec{r}\,'\left(t\right)\right|dt$$

# Line Integral (2 theorems)

- 1) Let *f* be continuous function along curve *C* (defined as before):

$$\oint_C f\left(x,y\right)ds=\int_a^b f\left(x\left(t\right),y\left(t\right)\right)\frac{ds}{dt}dt=\int_a^b f\left(x\left(t\right),y\left(t\right)\right)\sqrt{x_t^2+y_t^2}\,dt \qquad \text{or}$$

$$\oint_C f\left(x,y,z\right)ds=\int_a^b f\left(x\left(t\right),y\left(t\right),z\left(t\right)\right)\frac{ds}{dt}dt=\int_a^b f\left(x\left(t\right),y\left(t\right)\right)\sqrt{x_t^2+y_t^2+z_t^2}\,dt \qquad \text{or}$$

$$\oint_C f\left(\vec{x}\right)ds=\int_a^b f\left(\vec{r}\left(t\right)\right)\left|\vec{r}\,'\left(t\right)\right|dt,\ \text{for either } \vec{x}=\left(x,y\right) \text{ or } \vec{x}=\left(x,y,z\right)$$

  regardless of the parameterization as long as the curve traversed exactly once between a and b.

- 2) *Let* $C=C_1\cup C_2\cup\ldots$ *be piecewise smooth curve, then* $\oint_C f\,ds=\oint_{C_1} f\,ds+\oint_{C_2} f\,ds+\ldots$

# Line Integral(examples)

- 1) Let $\vec{r}\left(t\right)=\left\langle 2\cos t, 2\sin t\right\rangle, 0\le t\le \pi$ (half circle, r=2), evaluate $\oint\limits_C x^2 y ds$

  **Solution**:
  We have $f\left(x,y\right)=x^2 y \Rightarrow f\left(\vec{r}\left(t\right)\right)=f\left(2\cos t, 2\sin t\right)=4\cos^2 t\cdot 2\sin t=8\cos^2 t\cdot\sin t$
  the arclength is $\left|\vec{r}\,'\left(t\right)\right|=2\left|\left\langle -\sin t,\cos t\right\rangle\right|=2$
  thus

$$\oint\limits_C x^2 y ds=\int\limits_0^{\pi}\underbrace{8\cos^2 t\cdot\sin t}_{f\left(\vec{r}\left(t\right)\right)}\cdot\underbrace{2}_{\left|\vec{r}\,'\left(t\right)\right|}dt=16\int\limits_0^{\pi}\cos^2 t\cdot\sin t dt\underset{u=\cos t}{=}16\int\limits_{-1}^{1}u^2 du=\frac{32}{3}$$

# Line Integral(examples)

- 2) Let $C_1 = r_1(t) = \left\langle \sqrt{8}t, 4t, 5t \right\rangle, 0 \le t \le 1$ and $C_2 = r_2(t) = \left\langle 1, 2, -5t \right\rangle, -1 \le t \le 0$ evaluate $\oint\limits_{C_1 \cup C_2} x + y + zds$

**Solution**:

$$\oint\limits_{C_1} x + y + zds = \int_0^1 \underbrace{\left(\sqrt{8}t + 4t + 5t\right)}_{f(\vec{r}(t))} \cdot \underbrace{\sqrt{8+16+25}}_{|\vec{r}\,'(t)|}\, dt = 7\left(9+\sqrt{8}\right)\int_0^1 tdt = 7\left(9+\sqrt{8}\right)\frac{t^2}{2}\bigg|_0^1 = \frac{7\left(9+\sqrt{8}\right)}{2}$$

$$\oint\limits_{C_2} x + y + zds = \int_{-1}^0 \underbrace{\left(1+2-5t\right)}_{f(\vec{r}(t))} \cdot \underbrace{\sqrt{0+0+25}}_{|\vec{r}\,'(t)|}\, dt = 5\int_{-1}^0 3 - 5tdt = 5\left(3t - 5\cdot\frac{t^2}{2}\right)_{-1}^{0} = -5\left(-3-5\cdot\frac{1}{2}\right) = 15 + \frac{25}{2} = \frac{55}{2}$$

$$\oint\limits_{C_1 \cup C_2} x + y + zds = \oint\limits_{C_1} x + y + zds + \oint\limits_{C_2} x + y + zds = \frac{7\left(9+\sqrt{8}\right)}{2} + \frac{55}{2} = 59 + \frac{7\sqrt{8}}{2}$$

# Line Integral(examples)

- 3) Evaluate $\oint_C \frac{xy}{\sqrt{13}}\,ds$ where *C* is a line between (1,2) and (3,-1)

**Solution**: parameterize $\vec{r}(t) = (1-t)\langle 1,2\rangle + t\langle 3,-1\rangle = \langle 1+2t, 2-3t\rangle$

to get $\oint_C \frac{xy}{\sqrt{13}}\,ds = \int_0^1 \underbrace{\frac{(1+2t)(2-3t)}{\cancel{\sqrt{13}}}}_{f(\vec{r}(t))} \cdot \underbrace{\cancel{\sqrt{4+9}}}_{|\vec{r}\,'(t)|}\,dt = \int_0^1 2+t-6t^2 dt = \frac{1}{2}$

**Alternative Solution**: parameterize $\vec{r}(t) = (1-t)\langle 3,-1\rangle + t\langle 1,2\rangle = \langle 3-2t, -1+3t\rangle$

to get $\oint_C \frac{xy}{\sqrt{13}}\,ds = \int_0^1 \underbrace{\frac{(3-2t)(-1+3t)}{\cancel{\sqrt{13}}}}_{f(\vec{r}(t))} \cdot \underbrace{\cancel{\sqrt{4+9}}}_{|\vec{r}\,'(t)|}\,dt = \int_0^1 -3+11t-6t^2 dt = \frac{1}{2}$

# Line Integral(theorem + definition)

- In previous example we traversed a curve (line) in two opposite direction and got the same result – it didn't happen by an accident.
- **Theorem**: Denote by *-C* the same curve as *C*, but with different direction: $\oint_C f\,ds = \oint_{-C} f\,ds$
- **Definition**: Denote $\oint_C f\,ds$ a line integral with respect to arclength, a line integral with respect to x: $\oint_C f(x,y)\,dx = \int_a^b f\big(x(t),y(t)\big)x'(t)\,dt$, with respect to y: $\oint_C f(x,y)\,dy = \int_a^b f\big(x(t),y(t)\big)y'(t)\,dt$ and analogically in 3D $\oint_C f(x,y,z)\,dx, \oint_C f(x,y,z)\,dy, \oint_C f(x,y,z)\,dz$. They often occur together, e.g.

$$\oint_C f(x,y)\,dx + \oint_C g(x,y)\,dy = \oint_C f(x,y)\,dx + g(x,y)\,dy$$

# Line Integral(example)

- Evaluate $\oint\limits_{\substack{\langle 2+t,4t,5t\rangle, \\ 0\le t\le 1}} ydx+zdy+xdz$

**Solution**:

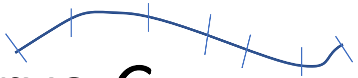
$$\oint\limits_{\substack{\langle 2+t,4t,5t\rangle, \\ 0\le t\le 1}} ydx+zdy+xdz=\int_0^1 \underset{y}{4t}\underbrace{\frac{d}{dt}\left(2+t\right)}_{x'(t)}dt+\int_0^1 \underset{z}{5t}\underbrace{\frac{d}{dt}\left(4t\right)}_{y'(t)}dt+\int_0^1 \underbrace{\left(2+t\right)}_{x}\underbrace{\frac{d}{dt}\left(5t\right)}_{z'(t)}dt$$

$$=\int_0^1 4t+20t+5\left(2+t\right)dt=\int_0^1 29t+10dt=24.5$$

# Line Integral of Vector Field

- **Reminder**:
    - A work done by variable force *f(x)* in moving a particle from *a* to *b* along the *x*-axis is given by $W = \int_a^b f(x)\,dx$.
    - A work done by a constant force **F** in moving object from point P to point Q in space is $W = \mathbf{F} \cdot \overrightarrow{PQ}$.
    - Unit tangent vector: $\mathbf{T}(t) = \dfrac{\vec{r}\,'(t)}{\left|\vec{r}\,'(t)\right|}$

$\mathbf{F}$ is $\sim constant\ on\ s_i$



- Consider now a variable force **F**(*x,y,z*) along a smooth curve *C*.
    - Divide *C* into number of a small enough sub-arcs so that the force is roughly constant on each sub-arc.
    - The displacement vector becomes unit tangent (**T**) times displacement ($\Delta s_i$):

$$\overrightarrow{PQ} = \Delta s_j \mathbf{T}\left(x\left(t_i^*\right), y\left(t_i^*\right), z\left(t_i^*\right)\right), t_i^* \in \left[t_{i-1}, t_i\right]$$

# Line Integral of Vector Field(cont)

- Finally the work of **F**(*x,y,z*) along *C* is given by

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}\mathbf{F}\left(x\left(t_i^*\right), y\left(t_i^*\right), z\left(t_i^*\right)\right)\cdot\mathbf{T}\left(x\left(t_i^*\right), y\left(t_i^*\right), z\left(t_i^*\right)\right)\Delta s_j$$

$$= \oint_C \mathbf{F}\left(x, y, z\right)\cdot\mathbf{T}\left(x, y, z\right)ds = \oint_C \mathbf{F}\cdot\mathbf{T}ds$$

- Denote $d\mathbf{r} = \vec{r}\,'\left(t\right)$ or $d\vec{r} = \vec{r}\,'\left(t\right)$

$$\oint_C \mathbf{F}\cdot\mathbf{T}ds = \int_a^b \left(\mathbf{F}\left(\vec{r}\left(t\right)\right)\cdot\frac{\vec{r}\,'\left(t\right)}{\cancel{\left|\vec{r}\,'\left(t\right)\right|}}\right)\cancel{\left|\vec{r}\,'\left(t\right)\right|}dt = \int_a^b \mathbf{F}\left(\vec{r}\left(t\right)\right)\cdot\vec{r}\,'\left(t\right)dt \equiv \int_a^b \underbrace{\mathbf{F}}_{\mathbf{F}\left(\vec{r}\left(t\right)\right)}\cdot\underbrace{d\mathbf{r}}_{d\vec{r}\,'}$$

# Line Integral of Vector(example)

- Let VF be given by $\mathbf{F} = \langle x, x+y, x+y+z \rangle$ and
- the curve $C$ by $\vec{r}(t) = \langle \sin t, \cos t, \sin t + \cos t \rangle, 0 \le t \le 2\pi$
  Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

**Solution**:

$$\mathbf{F} \cdot d\mathbf{r} = \underbrace{\langle \sin t, \cos t + \sin t, 2(\sin t + \cos t) \rangle}_{\mathbf{F}(\vec{r}(t))} \cdot \underbrace{\langle \cos t, -\sin t, \cos t - \sin t \rangle dt}_{= r'(t) dt}$$

$$= \left(2\cos^2 t - 3\sin^2 t\right) dt = \left( \underbrace{1 + \cos 2t}_{=2\cos^2 t} - \frac{3}{2} \underbrace{(1 - \cos 2t)}_{=2\sin^2 t} \right) dt$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1 + \cos 2t - \frac{3}{2}(1 - \cos 2t)\, dt = \left( t + \frac{\sin 2t}{2} - \frac{3}{2}\left( t - \frac{\sin 2t}{2} \right) \right)_0^{2\pi} = -\pi$$

# Line integral of Vector vs Scalar fields

- Let $\mathbf{F}(x,y,z) = \left\langle P(x,y,z), Q(x,y,z), R(x,y,z) \right\rangle$
  and $\vec{r}(t) = \left\langle x(t), y(t), z(t) \right\rangle, a \le t \le b$
  then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}\left(\vec{r}(t)\right) \cdot \bar{r}'(t)\, dt$$

$$= \int_a^b \left\langle P(x,y,z), Q(x,y,z), R(x,y,z) \right\rangle \cdot \left\langle x'(t), y'(t), z'(t) \right\rangle dt$$

$$= \int_a^b P(x,y,z)\, x'(t) + Q(x,y,z)\, y'(t) + R(x,y,z)\, z'(t)\, dt$$

$$= \int_a^b P(x,y,z)\, dx + Q(x,y,z)\, dy + R(x,y,z)\, dz$$

# Fundamental Theorem for Line Integrals

- **Recall:** Fundamental Theorem of Calculus (FTC) $\int_a^b F'(x)\,dx = F(b) - F(a)$
- Definition: A vector field **F** is called a **conservative vector field** if there exist a **potential**, a function *f*, such that $\mathbf{F} = \vec{\nabla} f$.
- **Theorem**: Let *C* be a smooth curve given by $\vec{r}(t), a \le t \le b$. Let **F** be a *continuous conservative vector field*, and *f* is a differentiable function satisfying $\mathbf{F} = \vec{\nabla} f$. Then, .

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \nabla f \cdot d\mathbf{r} = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right)$$

# Fundamental Theorem for Line Integrals(cont)

- Proof:

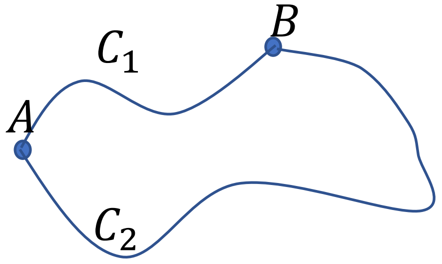
$$\oint_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f\left(\vec{r}(t)\right) \cdot \vec{r}\,'(t)\,dt = \int_a^b \left\langle f_x, f_y, f_z \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$= \int_a^b \left( f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \right) dt = \int_a^b \frac{d}{dt} f\left(\vec{r}(t)\right) dt$$

$$= f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right)$$

# Fundamental Theorem for Line Integrals(cont)

- **Definition**: Let **F** be continuous on domain D. The integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is called **independent of path in D** if for *any two* curves $C_1$, $C_2$ with the same initial and end points, we have




$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$$

- **Corollary**: A line integral of a conservative vector field is independent of path.
- **Definition**: A curve *C* is called closed if its terminal points coincides.

# Fundamental Theorem for Line Integrals(cont)

- **Theorem**: The integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D **if and only if** $\oint_C \mathbf{F}\cdot d\mathbf{r}=0$ on any closed curve *C*.

**Proof**: (➔) Let $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D. Let *C* be *arbitrary* closed curve. Choose any two points on *C*, A and B. Let $C_1$ be the curve from A to B, and $C_2$ from B to A, so that $C=C_1\cup C_2$, then

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_{C_1} \mathbf{F}\cdot d\mathbf{r}+\oint_{C_2} \mathbf{F}\cdot d\mathbf{r}=\oint_{C_1} \mathbf{F}\cdot d\mathbf{r}-\oint_{-C_2} \mathbf{F}\cdot d\mathbf{r}=0$$

**(🡐)** Let $\oint_C \mathbf{F}\cdot d\mathbf{r}=0$ on any closed curve *C* in D. Choose A,B∈D and let $C_1$, $C_2$ be arbitrary paths from A to B. $C=-C_1\cup C_2$ is closed curve, thus

$$0=\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_{-C_1} \mathbf{F}\cdot d\mathbf{r}+\oint_{C_2} \mathbf{F}\cdot d\mathbf{r}=-\oint_{C_1} \mathbf{F}\cdot d\mathbf{r}+\oint_{C_2} \mathbf{F}\cdot d\mathbf{r}\Rightarrow\oint_{C_1} \mathbf{F}\cdot d\mathbf{r}=\oint_{C_2} \mathbf{F}\cdot d\mathbf{r}$$

# Fundamental Theorem for Line Integrals(cont)

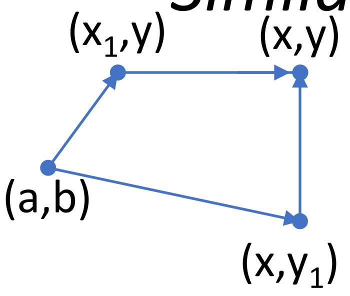
- **Theorem**: Suppose **F**=<P,Q> is continuous vector field on an open connected region D. If $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D, then **F** is conservative vector field in D, that is there is *f* such that $\mathbf{F} = \vec{\nabla} f$.

**Proof:** Let (*a,b*)∈D be arbitrary fixed point. Define $f(x,y) = \int_{(a,b)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r}$ .

Due to independency of path we can choose path *C* from (*a*,*b*) to (*x*,*y*) that crosses ($x_1$,*y*)∈D, $x_1$ is const.

$$f_x(x,y) = \frac{d}{dx}\int_{(a,b)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r} = \cancelto{=0,\text{ no } x}{\frac{d}{dx}\int_{(a,b)}^{(x_1,y)} \mathbf{F}\cdot d\mathbf{r}} + \frac{d}{dx}\int_{(x_1,y)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r} = \frac{d}{dx}\int_{(x_1,y)}^{(x,y)} Pdx + \cancelto{dy=0}{Qdy} = \frac{d}{dx}\int_{x_1}^{x} Pdx \underset{FTC}{=} P$$

*Similarly,*




$$f_y(x,y) = \frac{d}{dy}\int_{(a,b)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r} = \frac{d}{dy}\int_{(a,b)}^{(x,y_1)} \mathbf{F}\cdot d\mathbf{r} + \cancelto{=0,\text{ no y}}{\frac{d}{dy}\int_{(x,y_1)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r}} = \frac{d}{dy}\int_{(x,y_1)}^{(x,y)} \cancelto{dx=0}{Pdx} + Qdy = \frac{d}{dy}\int_{y_1}^{y} Qdy \underset{FTC}{=} Q$$

# Fundamental Theorem for Line Integrals(cont)

- **Theorem**: Suppose **F**=<P,Q> is a conservative vector field and P,Q has continuous first order partial derivatives on domain D, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

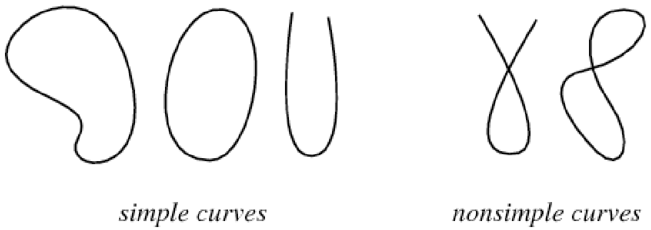
**Proof**: Let *f* be the potential, i.e. $\langle P, Q \rangle = \mathbf{F} = \vec{\nabla} f = \langle f_x, f_y \rangle$, therefore

$$f_{xy} = \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = f_{yx}$$

# Fundamental Theorem for Line Integrals(cont)

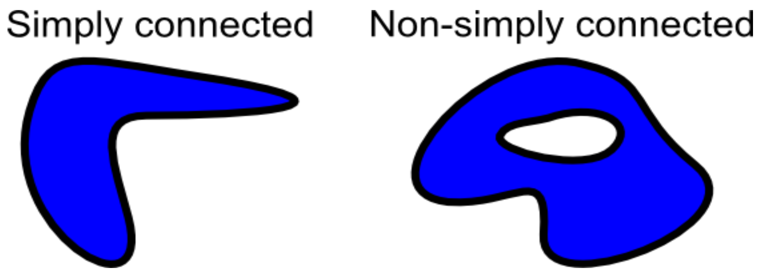
- **Definitions**:

1) A **simply connected curve** is a curve that doesn't intersect itself between endpoints.




2) A **simple closed curve** is a curve with $\vec{r}(a)=\vec{r}(b)$ but $\vec{r}(t_1)\neq\vec{r}(t_2)$ for any $a<t_1<t_2<b$.

3) A **simply connected region**: is a region D in which every simple closed curve encloses only points from D. In other words D consist of one piece and has no hole.



# Fundamental Theorem for Line Integrals(cont)

- **Theorem**: Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field on an open simply connected region D. If P,Q have continuous first order partial derivatives on domain D and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\mathbf{F}$ is conservative.

- **Example**: Determine whether $\mathbf{F}(x, y) = \langle x \sin y, y \sin x \rangle$ is conservative.
  **Solution**: Not conservative, since

$$P_y = (x \sin y)_y = x \cos y \neq y \cos x = (y \sin x)_x = Q_x$$

# Fundamental Theorem for Line Integrals(cont)

- **Example**: Show that $\mathbf{F}(x,y)=\langle P,Q\rangle=\langle x+y,x-y\rangle$ and find the potential.

**Solution**: $P_y=(x+y)_y=1=(x-y)_x=Q_x$ , indeed **F** is conservative.

- To find the potential start with

$$f(x,y)=\int f_x(x,y)\,dx=\int x+y\,dx=\frac{x^2}{2}+yx+g(y)$$

  note that the constant of integration can be function of y.

- To find *g* differentiate and compare to *Q*: $f_y=x+g'(y)=x-y$

  *to get* $g(y)=\int g'(y)\,dx=-\int y\,dx=-\frac{y^2}{2}+const$

- Finally, since any potential works, set *const=0 to get*

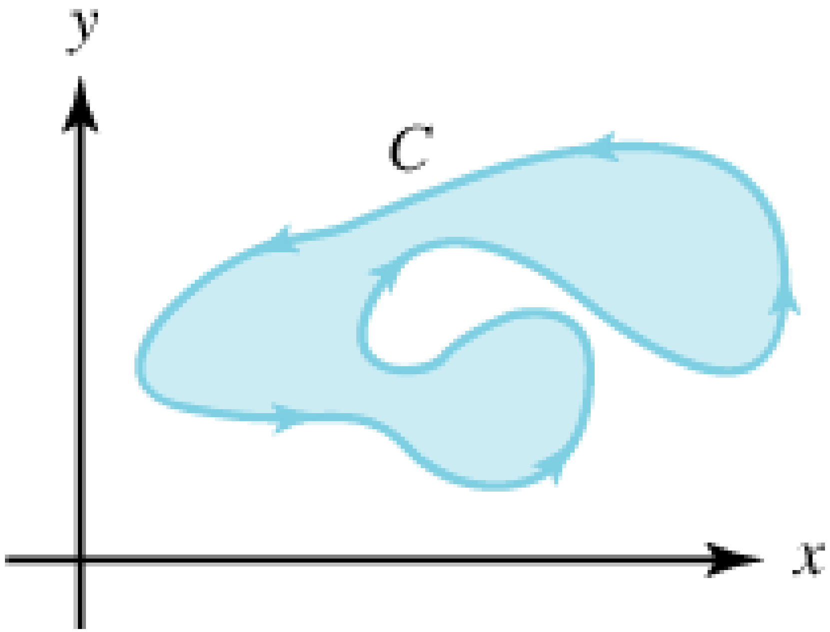
$$f(x,y)=\frac{x^2}{2}+yx-\frac{y^2}{2}$$

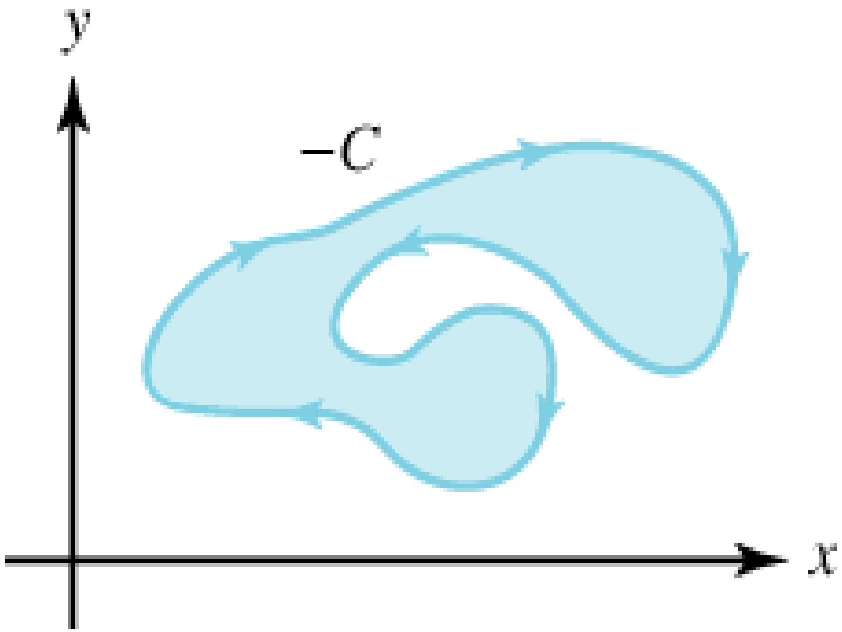
# Green’s Theorem

- **Definition**: A simple closed curve is said to be **positive oriented** if it traversed **counterclockwise**.

Counterclockwise – positively oriented

Clockwise – negatively oriented



# Green’s Theorem(the theorem)

- **Green’s Theorem**: Let *C* be **positively oriented piecewise-smooth, simple closed curve** in the plane and let ***D*** be the **region bounded by *C***. If *P* and *Q* have continuous partial derivatives on an open region that contains *D*, then

$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

- **Note**: The circle on the line integral ($\oint$) is sometime related to the positive oriented curve and sometime even drawn with an arrow on the circle: $\oint$

# Green’s Theorem(cont)

- One views the Green’s theorem as a counterpart of Fundamental Theorem of Calculus
  - Green’s theorem $$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \oint_{\partial D} P\,dx + Q\,dy$$
  - FTC theorem $$\int_a^b F'(x)\,dx = F(b) - F(a)$$
- Notice that in both, the left side is on the domain while the right one is at the boundary of the domain.

# Green’s Theorem(proof)

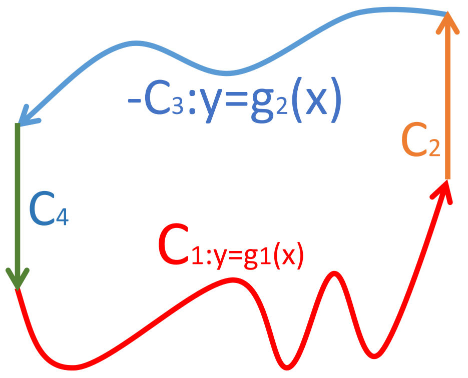
**Proof**:

- Formulate *D* as domain of type I and show that $\oint_{\partial D} P dx = -\iint_D \frac{\partial P}{\partial y} dA$

  thus, let $D = \{a \le x \le b, g_1(x) \le y \le g_2(x)\}$
  and let $C = C_1 \cup C_2 \cup C_3 \cup C_4$, as depicted

$$\oint_{\partial D} P dx = \oint_{\langle x, g_1(x)\rangle} P dx + \oint_{\langle b, g_1(b)\rangle(1-t) + \langle b, g_2(b)\rangle t} P dx - \oint_{\langle x, g_2(x)\rangle} P dx + \oint_{\langle a, g_2(a)\rangle(1-t) + \langle a, g_1(a)\rangle t} P dx$$

$$= \int_a^b P(x, g_1(x))dx + \cancel{\int_b^b P dx} - \int_a^b P(x, g_2(x))dx + \cancel{\int_a^a P dx}$$




which is the same as $-\iint_D \frac{\partial P}{\partial y} dA = -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} dy dx = \int_a^b \int_{g_2(x)}^{g_1(x)} \frac{\partial P}{\partial y} dy dx = \int_a^b P(x, g_1(x)) - P(x, g_2(x)) dx$

- Similarly, one formulate *D* as domain of type II to show that $\oint_{\partial D} Q dy = \iint_D \frac{\partial Q}{\partial x} dA$

# Green's Theorem(cont)

- Example: Let D be square $[0,2]\times[0,2]$. Evaluate $\oint_{\partial D}\left(x^2-xy^3\right)dx+\left(y^2-2xy\right)dy$
  **Solution**: Using Green's theorem,

$$\oint_{\partial D}\underbrace{\left(x^2-xy^3\right)}_{P}dx+\underbrace{\left(y^2-2xy\right)}_{Q}dy=\iint_D\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}dA=\iint_D\frac{\partial}{\partial x}\left(y^2-2xy\right)-\frac{\partial}{\partial y}\left(x^2-xy^3\right)dA$$

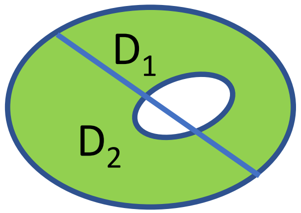
$$=\int_0^2\int_0^2-2y+3xy^2dxdy=\int_0^2\left(-2xy+3\frac{x^2}{2}y^2\right)_0^2dy=\int_0^2\left(-4y+3\frac{2^{\cancel{2}}}{\cancel{2}}y^2\right)_0^2dy=\left(-2y^2+2y^3\right)_0^2=8$$

- Verify $\oint_{\partial D}\left(x^2-xy^3\right)dx+\left(y^2-2xy\right)dy=\int_{\langle x,0\rangle}+\int_{\langle 2,y\rangle}-\int_{\langle x,2\rangle}-\int_{\langle 0,y\rangle}=$

$$=\int_0^2 x^2dx+\int_0^2 y^2-4ydy-\int_0^2 x^2-2^3xdx-\int_0^2 y^2dy=8\int_0^2 xdx-4\int_0^2 ydy=4x^2\Big|_0^2-2y^2\Big|_0^2=16-8=8$$

# Green's Theorem(extensions)

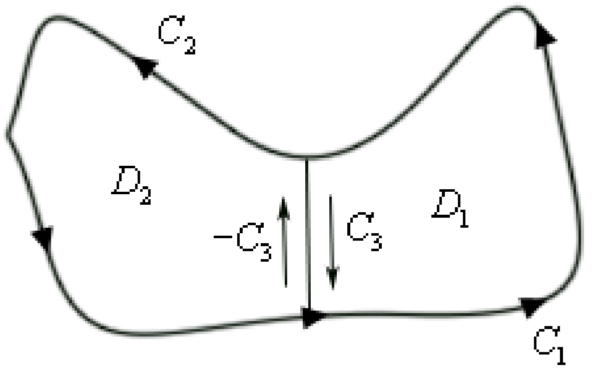
- How to use Green's theorem beyond its original formulation?
  - In the case when the curve $C$ is not closed (but its line integral isn't "nice"):
    - Connect the endpoints of $C$ with any **simple curve** $C_1$ to get $C_2 = C \cup C_1$
    - Now, $\int_{C_2}$ can conveniently(?) be evaluated using Green's theorem and $\int_C = \int_{C_2} - \int_{C_1}$
    - Hint: The best choice of $C_1$ will make $\int_{C_1}$ easy.
- In the case the region D has a hole, i.e. is not a simply connected.
    - Rewrite D as union of simply connected regions (see example)
    - Use the version of Green's theorem for Union of Domains (TBD on next slide)



# Green's Theorem(extensions)

- **Theorem**: Let *D* be a domain. Rewrite *D* as union of 2 subdomains, e.g. $D = D_1 \cup D_2$, let $\partial D = C_1 \cup C_2$ and $C_3 = D_1 \cap D_2$, such that $\partial D_1 = C_1 \cup C_3$ and $\partial D_2 = C_2 \cup (-C_3)$, then

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \oint_{C_1 \cup C_3} Pdx + Qdy + \oint_{C_2 \cup (-C_3)} Pdx + Qdy$$




$$= \oint_{C_1} Pdx + Qdy + \oint_{C_2} Pdx + Qdy = \oint_{C_1 \cup C_2} Pdx + Qdy$$

# Green’s Theorem(extensions)

- **Example**: Evaluate $A = \iint_D dA$.
- **Solution**: For a smart use of Green’s Theorem: choose any *P* and *Q*, such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.
  - For example $P = 0, Q = x$, gives

$$A = \iint_D dA = \oint_{\partial D} x dy$$

# Green's Theorem(extensions)

- Let C: $\vec{r}(t)=\left\langle t,\sqrt{t-t^2}\right\rangle, t\in[0,1]$. Evaluate: $\oint_C \underbrace{\left(e^x\sin y-y^2+x\right)}_{P}dx+\underbrace{\left(e^x\cos y-\cos y^2\right)}_{Q}dy$
- **Solution**: reformulate the curve as $y=\sqrt{x-x^2}$ or $y^2+x^2=x$ which is a half circle, or in polar coordinates $r=\cos\theta, 0\le\theta\le\pi/2$. Connect the ends of the half circle with a line along x-axis, from 0 to 1.

$$\oint_C Pdx+Qdy=\oint_{C\cup C_1}Pdx+Qdy-\oint_{C_1}Pdx+Qdy=\iint_R Q_x-P_y dA-\oint_{\langle x,0\rangle}Pdx+Qdy=\frac{1}{6}-\frac{1}{2}=-\frac{1}{3}$$

$$\iint_R Q_x-P_y dA=\iint_R e^x\cos y-\left(e^x\cos y-2y\right)dA=\iint_R 2ydA=\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta}2r\sin\theta\cdot rdrd\theta=\frac{1}{6}$$

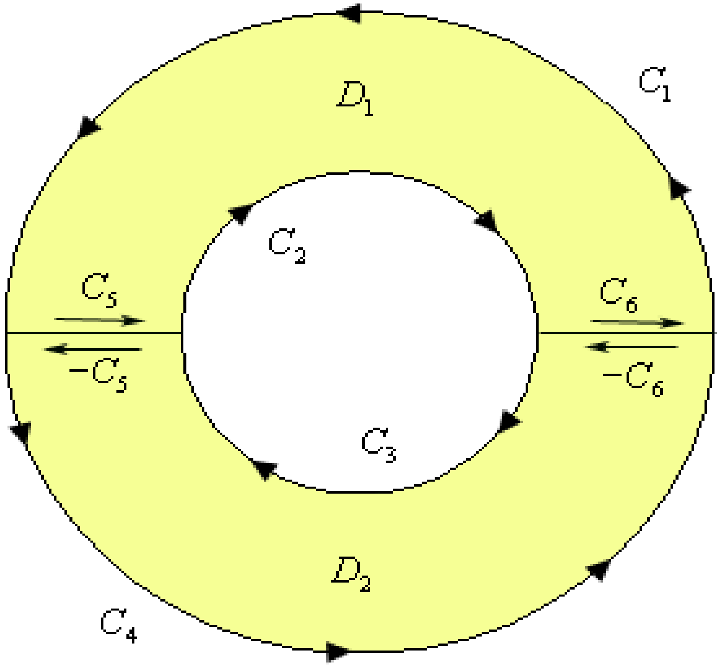
$$\oint_{\langle x,0\rangle}Pdx+\cancel{Qdy}^{=0}=\int_0^1\left(e^t\sin 0-0^2+t\right)\frac{d}{dt}tdt=\left.\frac{t^2}{2}\right|_0^1=\frac{1}{2}$$

# Green's Theorem(extensions)

- **Example**: Let *C* be a ring with radiuses 1 and 2 centered at the origin.

$$\oint_C \underset{P}{y^3}\, dx + \underset{Q}{x^3}\, dy = \iint_{D_1} \underset{Q_x}{-3x^2} - \underset{P_y}{3y^2}\, dA + \iint_{D_2} \underset{Q_x}{-3x^2} - \underset{P_y}{3y^2}\, dA$$

$$= -3\int_0^{2\pi}\int_1^2 r^3 dr d\theta = -3\cdot 2\pi \left.\frac{r^4}{4}\right|_1^2 = -\frac{45}{2}\pi$$



# Curl and Divergence

- Let **F=**<*P,Q,R*> be a vector field on $\mathbb{R}^3$. Assume that all partial derivatives of *P*,*Q*,*R* exists, then
  - the curl of **F** is defined as

$$\operatorname{curl}\mathbf{F} = \vec{\nabla}\times\mathbf{F} = \det\begin{pmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

  - the divergence of **F** is defined as

$$\operatorname{div}\mathbf{F} = \vec{\nabla}\cdot\mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R\rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

# Curl and Divergence(cont)

- Example: Let $f(x,y,z) = x\sin yz$. Then $\mathbf{F} = \vec{\nabla} f = \langle \sin yz, xz\cos yz, xy\cos yz\rangle$,

$$\begin{aligned} \operatorname{div}\mathbf{F} &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle \sin yz, xz\cos yz, xy\cos yz\rangle \\ &= 0 + \left(-xz^2 \sin yz\right) + \left(-xy^2 \sin yz\right) = -x\left(y^2 + z^2\right)\sin yz \end{aligned}$$

and $\operatorname{curl}\mathbf{F} = 0$ since

$$\frac{\partial R}{\partial y} = x\cos yz - xyz\sin yz = \frac{\partial Q}{\partial z},$$

$$\frac{\partial P}{\partial z} = y\cos yz = \frac{\partial R}{\partial x}, \text{and}$$

$$\frac{\partial Q}{\partial x} = z\cos yz = \frac{\partial P}{\partial y}$$

# Curl and Divergence (cont)

- **Theorem**: Suppose f(x,y,z) has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = curl\langle f_x, f_y, f_z\rangle = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy}\rangle = 0$$

- **Theorem**: If **F** is vector field defined on $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl}\mathbf{F} = 0$, then **F** is a conservative vector field.
- **Theorem**: Suppose **F=<*P,Q,R*>** is a vector field on $\mathbb{R}^3$ and has continuous second-order partial derivatives, then

$$\operatorname{div}\operatorname{curl}\mathbf{F} = \operatorname{div}\left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle = \left(\frac{\partial^2 R}{\partial x\partial y} - \frac{\partial^2 Q}{\partial x\partial z}\right) + \left(\frac{\partial^2 P}{\partial y\partial z} - \frac{\partial^2 R}{\partial y\partial x}\right) + \left(\frac{\partial^2 Q}{\partial z\partial x} - \frac{\partial^2 P}{\partial z\partial y}\right) = 0$$

# Curl and Divergence (cont)

- *Recall:* $\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial D} \mathbf{F} \cdot \mathbf{T} ds$, *where* $\mathbf{T}$ is the unit tangent vector.
- **Green's theorem in vector form (Tangential Component)**: Let $\mathbf{F} = \langle P(x,y), Q(x,y), 0 \rangle$, *then*

$$\iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} dA = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

- *Proof:*

$$\operatorname{curl} \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, 0 \rangle = \left\langle 0 - \underbrace{\frac{\partial Q}{\partial z}}_{=0}, \underbrace{\frac{\partial P}{\partial z}}_{=0} - 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

$$\iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} dA = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \underbrace{\mathbf{k} \cdot \mathbf{k}}_{=1} dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \oint_{\partial D} P dx + Q dy = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

# Curl and Divergence (cont)

- **Recall: unit normal vector is given by** $\mathbf{n}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \overset{\text{in } 2D,\ \text{verify!}}{=} \dfrac{\langle y'(t), -x'(t)\rangle}{|\vec{r}\,'(t)|}$
- **Green's theorem in vector form (Normal Component)**: Let **F**=<*P,Q*> be a vector field and $\vec{r}(t) = \langle x(t), y(t)\rangle$, then

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}(x,y)\,dA$$

- **Proof**:

$$\mathbf{F}(\vec{r}(t))\cdot\mathbf{n} = \langle P(x(t),y(t)), Q(x(t),y(t))\rangle \cdot \frac{\langle y'(t), -x'(t)\rangle}{|\vec{r}\,'(t)|} = \frac{Py'(t) - Qx'(t)}{|\vec{r}\,'(t)|}$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_a^b \frac{Py'(t)+Qd'(t)}{\cancel{|\vec{r}\,'(t)|}} \cancel{|\vec{r}\,'(t)|}\,dt = \int_a^b P\,dy - Q\,dx$$

$$= \int_a^b (-Q)\,dx + P\,dy = \iint_D \frac{\partial P}{\partial x} - \frac{\partial(-Q)}{\partial y}\,dA \underset{Green's}{=} \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\,dA = \iint_D \operatorname{div}\mathbf{F}(x,y)\,dA$$

# Curl and Divergence (cont)

- **Example**: Evaluate $\iint_D y^2 - x^2 dA$, where *D* is unit disk.
- **Solution**: Previously we would evaluate it directly using a trigonometric identity

$$\iint_D y^2 - x^2 dA = \int_0^{2\pi}\int_0^1 r^2\left(\sin^2\theta - \cos^2\theta\right) r\,dr\,d\theta = -\int_0^{2\pi}\int_0^1 r^3\cos 2\theta\,dr\,d\theta = -\left(\frac{r^4}{4}\right)_0^1\left(\frac{\sin 2\theta}{2}\right)_0^{2\pi} = -\frac{1}{4}\cdot 0 = 0$$

- Now we can do something else, let $\vec{r}(t) = \langle \cos t, \sin t\rangle$ and $\mathbf{F} = \langle x^2 y, xy^2, 0\rangle$, then $y^2 - x^2 = (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}$, and

$$\iint_D y^2 - x^2 dA = \oint_{\partial D}\mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\underbrace{\left\langle \cos^2 t\sin t, \cos t\sin^2\theta, 0\right\rangle}_{\mathbf{F}(\vec{r}(t))}\cdot\underbrace{\left\langle \sin t, -\cos t, 0\right\rangle}_{\vec{r}\,'(t)}dt = \int_0^{2\pi}\cos^2 t\sin^2 t - \cos^2 t\sin^2 t\,dt = 0$$

- Furthermore, notice that **F** is conservative VF (since $\mathbf{F} = \vec{\nabla} x^2 y^2$), therefore vanishes on every closed curve, i.e. no integration required here.

# Curl and Divergence (cont)

- Example: Evaluate $\oint_C \frac{x^2y^2}{\sqrt{x^2+y^2}}ds$, *where C is a circle of radius* $\sqrt{2}$*.*
- **Solution** 1:

$$\oint_C \frac{x^2y^2}{\sqrt{x^2+y^2}}ds = \int_0^{2\pi} \frac{2\cos^2 t \cdot 2\sin^2 t}{\cancel{\sqrt{2\cos^2 t+2\sin^2 t}}}\cancel{\left|\vec{r}\,'(t)\right|}dt = \int_0^{2\pi} 4\cos^2 t\sin^2 t dt = \int_0^{2\pi} \sin^2 2t dt = \left(\tfrac{t}{2}-\tfrac{1}{8}\sin 4t\right)_0^{2\pi} = \pi$$

- **Solution** 2: Notice that normal to *C* is $\mathbf{n}(t) = \frac{\langle x,y\rangle}{\left|\langle x,y\rangle\right|}$ and $\frac{x^2y^2}{\sqrt{x^2+y^2}} = \underbrace{\frac{1}{2}\left\langle xy^2, x^2y\right\rangle}_{=\mathbf{F}} \cdot \underbrace{\frac{\langle x,y\rangle}{\left|\langle x,y\rangle\right|}}_{\mathbf{n}}$

  thus

$$\oint_C \mathbf{F}\cdot\mathbf{n}ds = \iint_D \operatorname{div}\mathbf{F}(x,y)dA = \frac{1}{2}\iint_D y^2 + x^2 dA = \frac{1}{2}\int_0^{2\pi}\int_0^{\sqrt{2}} r^2 \cdot r dr d\theta = \frac{1}{2}\cdot 2\pi\cdot\left(\frac{r^4}{4}\right)_0^{\sqrt{2}} = \pi$$

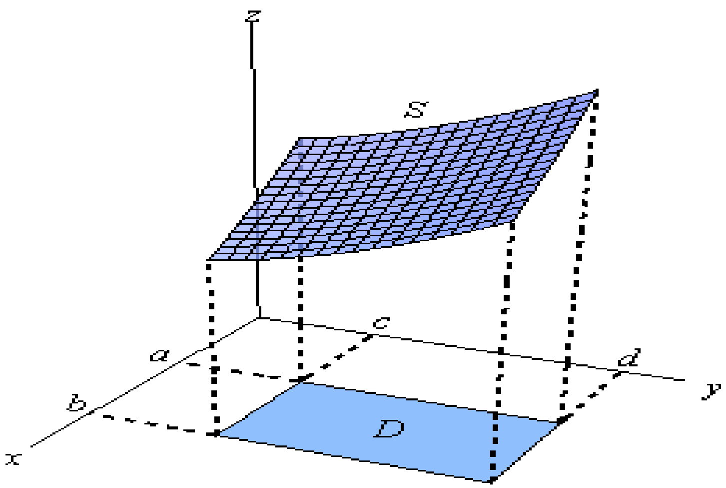
# Surface Integral

- **Definition**: Let $S$ be surface $\vec{r}(u,v)=\langle x(u,v),y(u,v),z(u,v)\rangle,(u,v)\in D$. Let $P_{ij}^*$ be a sample point on a patch $S_{ij}$ which area is given by $\Delta S_{ij}=|\vec{r}_u\times\vec{r}_v|\Delta u\Delta v$ .
  - $S_{ij}$ is defined by $[u_{i-1},u_i]\times[v_{j-1},v_j]$, consequently

$$P_{ij}^*=\left\langle x\left(u_i^*,v_j^*\right),y\left(u_i^*,v_j^*\right),z\left(u_i^*,v_j^*\right)\right\rangle,u_i^*\in[u_{i-1},u_1],v_j^*\in\left[v_{j-1},v_j\right]$$

The surface integral is given by

$$\oiint_S f(x,y,z)\,dS=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}f\left(P_{ij}^*\right)\Delta S_{ij}$$



# Surface Integral(cont)

- **Theorem**: $\oiint_S f(x,y,z)\,dS = \iint_D f(r(u,v))|r_u \times r_v|\,dA$

- Note that the area of the surface *S* is $A(S) = \oiint_S 1\,dS = \iint_D |r_u \times r_v|\,dA$
- **Reminder**: $\oint_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|\,dt$
- Note: The relationship of surface integral and surface area are analogical to the relationship between the line integral and arclength.

- Example:

$$\iint_{\partial D: z=g(x,y)} f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{(g_x)^2 + (g_y)^2 + 1}\,dA$$

# Surface Integral(cont)

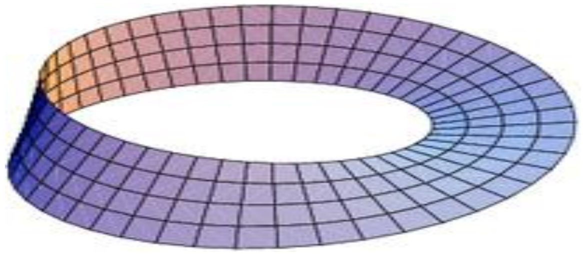
- **Example**: Let *S* be a unit sphere. The parametric representation is given by $\vec{r}(\varphi,\theta)=\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle, 0\le\varphi\le\pi, 0\le\theta\le 2\pi$

$$\begin{aligned}\left|\vec{r}_\varphi\times\vec{r}_\theta\right| &=\left|\langle\cos\phi\cos\theta,\cos\phi\sin\theta,-\sin\phi\rangle\times\langle-\sin\phi\sin\theta,\sin\phi\cos\theta,0\rangle\right|\\ &=\left|\langle\sin^2\phi\cos\theta,\sin^2\phi\sin\theta,\cos\phi\sin\phi\rangle\right|=\sin\varphi\end{aligned}$$

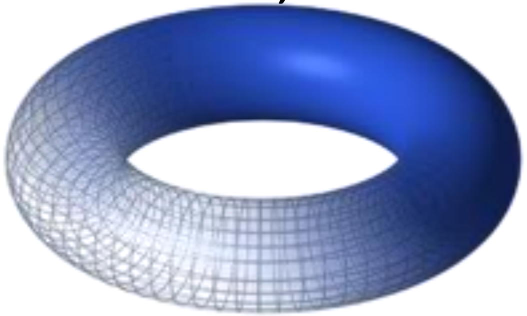
$$\begin{aligned}\oiint_S x^2 dS &= \iint_D \sin^2\varphi\cos^2\theta\left|\vec{r}_\varphi\times\vec{r}_\theta\right|dA=\iint_D \sin^3\varphi\cos^2\theta dA\\ &=\int_0^{2\pi}\cos^2\theta d\theta\int_0^{\pi}\sin^3\varphi d\varphi=\int_0^{2\pi}\frac{1+\cos 2\theta}{2}d\theta\int_0^{\pi}\left(\sin\varphi-\sin\varphi\cos^2\varphi\right)\varphi d\varphi\\ &=\frac{1}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)_0^{2\pi}\left(-\cos\varphi+\frac{1}{3}\cos^3\varphi\right)_0^{\pi}=\frac{4\pi}{3}\end{aligned}$$

# Surface Integral (cont)

- **Definition**: A two-sided surface *S* is called oriented if the unit normal vector **n** is defined at every point (except the boundary points). The orientation is chosen by direction (positive or negative) of the unit normal, in other words by choosing the side.

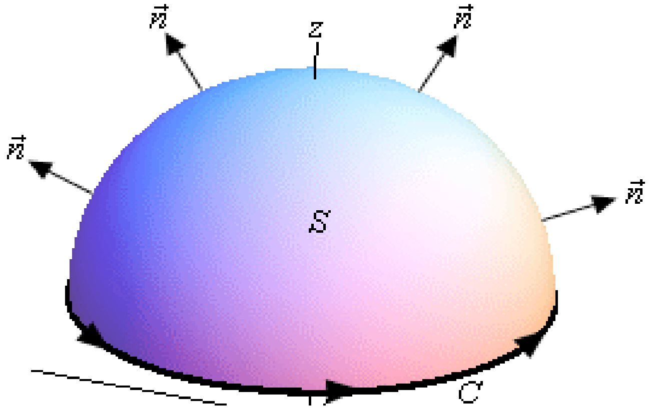


- **Example**: A Mobius strip is one-sided surface, therefore non orientable.



- A torus is two-sided, so it can be oriented inward or outward.

- **Definition**: A closed surface is a boundary of solid region. A closed surface is considered positive oriented if the unit normal points outward.



# Surface Integral (cont)

- **Definition**: Flux of **F** across surface S is defined by $\oiint_S \mathbf{F}\cdot d\mathbf{S} = \oiint_S \mathbf{F}\cdot\mathbf{n}\,dS$
- Notice the difference between $d\mathbf{S}$ and $dS$, and the similarity with $d\mathbf{r}$.
- Evaluation:

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \oiint_S \mathbf{F}\cdot\mathbf{n}\,dS = \oiint_S \mathbf{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{\left|\vec{r}_u\times\vec{r}_v\right|}\,dS$$

$$= \iint_D \left(\mathbf{F}\left(\vec{r}\left(u,v\right)\right)\cdot\frac{\vec{r}_u\times\vec{r}_v}{\left|\vec{r}_u\times\vec{r}_v\right|}\right)\left|\vec{r}_u\times\vec{r}_v\right|dA = \iint_D \mathbf{F}\cdot\left(\vec{r}_u\times\vec{r}_v\right)dA$$

# Surface Integral (cont)

- $\iint\limits_{\partial D: z=g(x,y)} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{D} \langle P,Q,R \rangle \cdot \langle -g_x, -g_y, 1 \rangle dA = \iint\limits_{D} -Pg_x - Qg_y + R dA$
- Rate of flow of a fluid with density $\rho$ and velocity field $\vec{v}$: $\oiint_S \rho \mathbf{v} \cdot \mathbf{n} dS$
- Electric flux of an electric field **E** through the surface *S*: $\oiint_S \mathbf{E} \cdot d\mathbf{S}$.
- **Gauss's Law**: a net charge enclosed by a closed surface *S:* $Q = \varepsilon_0 \oiint_S \mathbf{E} \cdot d\mathbf{S}$ *where* $\varepsilon_0$ is permittivity of free space.
- Let K be a conductivity constant of a substance and let u(*x*,*y*,*z*) denote temperature of a body. The of heat flow is given by $-K\nabla u$ and the rate of heat flow by $-K\oiint\limits_{S} \nabla u \cdot d\mathbf{S}$.

# Surface Integral(cont)

- **Example**: Let **F=<*x,y,z*>** *and* $\vec{r}\left(\varphi,\theta\right)=\left\langle \sin\varphi\cos\theta,\sin\varphi\sin\theta,\cos\varphi\right\rangle, 0\le\varphi\le\pi, 0\le\theta\le 2\pi$

$$\begin{aligned}\vec{r}_\varphi\times\vec{r}_\theta &= \left\langle \cos\phi\cos\theta,\cos\phi\sin\theta,-\sin\phi\right\rangle\times\left\langle -\sin\phi\sin\theta,\sin\phi\cos\theta,0\right\rangle \\ &= \left\langle \sin^2\phi\cos\theta,\sin^2\phi\sin\theta,\cos\phi\sin\phi\right\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{F}\left(\vec{r}\left(\varphi,\theta\right)\right)\cdot\left(\vec{r}_\varphi\times\vec{r}_\theta\right) &= \vec{r}\cdot\left(\vec{r}_\varphi\times\vec{r}_\theta\right) \\ &= \sin^2\phi\cos\phi\cos^2\theta+\sin^2\phi\cos\phi\sin^2\theta-\sin^2\phi\cos\phi \\ &= \sin\varphi\end{aligned}$$

$$\oiint_C \mathbf{F}\cdot d\mathbf{S}=\iint_D \mathbf{F}\left(\vec{r}\left(\varphi,\theta\right)\right)\cdot\left(\vec{r}_\varphi\times\vec{r}_\theta\right)dA=\iint_D \sin\varphi\, dA=\int_0^{2\pi}\int_0^{\pi}\sin\varphi\, d\varphi\, d\theta=-2\pi\left(\cos\varphi\right)_0^{\pi}=4\pi$$

# Stokes' Theorem

- Let *S* be an oriented piecewise-smooth surface that is bounded by a simple closed piecewise-smooth boundary curve *C* with positive orientation. Let **F** be a vector field whole components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains *S*. Then,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \oiint_{S} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

# Stokes’ Theorem(cont)

- Stokes’ Theorem says that the line integral around the boundary curve of *S* of the tangential component of **F** is equal to the surface integral over of the normal component of the curl of **F**.
- One of the important uses of Stoke’s Theorem is in calculating surface integrals over “non convenient” surface using surface integral over more convenient surface with the same boundary:

$$\oiint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{S_1 = \partial S = S_2} \mathbf{F} \cdot d\mathbf{r} = \oiint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

# Stokes' Theorem(cont)

- One see Stokes' Theorem as a sort of higher dimensional version of Green's theorem. Really, if *S* is flat and lies in xy plane, then **n=k** and therefore

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \oiint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oiint_S \left(\operatorname{curl} \mathbf{F}\right) \cdot \mathbf{k}\, dS$$

which is a vector form of Green's theorem.

- Thus, Green's theorem is a private case of Stokes Theorem.

# Stokes' Theorem(cont)

- **Proof** (of the light version): We restrict our proof only for the case of S given as z=g(x,y).

$$\oint_{\partial S} \mathbf{F}\cdot d\mathbf{r} = \int_a^b P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \int_a^b Pdx + Qdy + R\left(z_x dx + z_y dy\right)$$

$$= \int_a^b \left(P + Rz_x\right)dx + \left(Q + Rz_y\right)dy \underset{Green's}{=} \iint_D \frac{\partial}{\partial x}\left(Q + Rz_y\right) - \frac{\partial}{\partial y}\left(P + Rz_x\right)dA$$

$$\begin{pmatrix} f(x) = g(x, y(x)) \Rightarrow \\ f' = g_x + g_y y' \end{pmatrix}$$

$$= \iint_D \left(Q_x + Q_z z_x + \left(R_x + \cancel{R_z z_x}\right)z_y + \cancel{Rz_{yx}}\right) - \left(P_y + P_z z_y + \left(R_y + \cancel{R_z z_y}\right)z_x + \cancel{Rz_{xy}}\right)dA$$

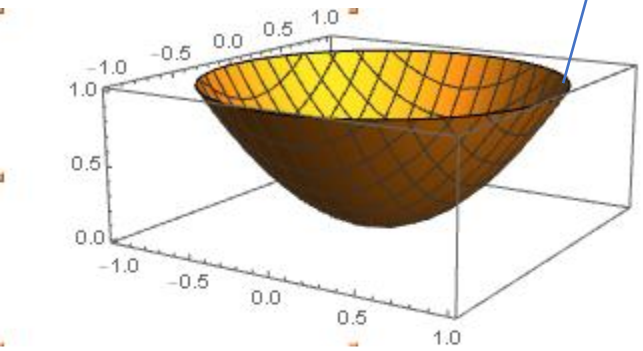
$$= \iint_D -\left(R_y - Q_z\right)z_x - \left(P_z - R_x\right)z_y + \left(Q_x - P_y\right)dA$$

$$= \iint_D \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \cdot \left\langle -z_x, -z_y, 1\right\rangle dA = \iint_D \operatorname{curl}\mathbf{F}\cdot\left(\vec{r}_u \times \vec{r}_v\right)dA = \oiint_{\partial D: z = g(x,y)} \mathbf{F}\cdot d\mathbf{S}$$

# Stokes' Theorem(cont)

- **Example**: Verify Stokes' Theorem for $\mathbf{F} = \langle yz, xz, xy \rangle$ over S: $z = x^2 + y^2 \le 1$

$$\oint_{\langle \cos t, \sin t, 1 \rangle} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\left(\langle \cos t, \sin t, 1 \rangle\right) \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$




$$= \int_0^{2\pi} \langle \sin t, \cos t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} -\sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} \cos 2t \, dt = \left. \frac{\sin 2t}{2} \right|_0^{2\pi} = 0$$

- From the other side we have, $\mathbf{F} = \nabla xyz$, therefore $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S 0 \cdot d\mathbf{S} = 0$

# Stokes' Theorem(cont)

- **Example**: Evaluate

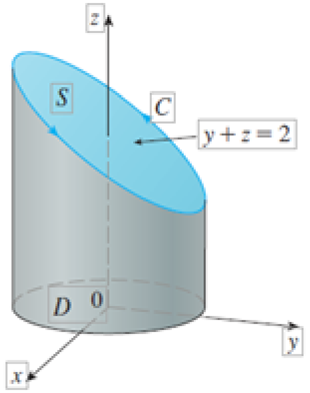
$$I = \oint_{\langle \cos t, \sin t, 2\rangle} \left(e^{-x^2/2} - yz\right)dx + \left(e^{-y^2/2} + xz + 2x\right)dy + \left(e^{-z^2/2} + 5\right)dz$$

- **Solution**: It is clear that a direct evaluation of the line integral is awkward. Therefore, denote $\mathbf{F} = \left\langle e^{-x^2/2} - yz, e^{-y^2/2} + xz + 2x, e^{-z^2/2} + 5 \right\rangle$, and use Stokes' Theorem. We also need $\operatorname{curl}\mathbf{F} = \langle x, -y, 2+2z\rangle$. Finally,

$$I \underset{Stokes}{=} \oiint_{disc} \langle x, -y, 2+2z\rangle \cdot \mathbf{n}dS \underset{\substack{\mathbf{n}=\mathbf{k} \\ z=2}}{=} 6\oiint_{disc} dA = 6A(disc) = 6\pi$$

# Stokes' Theorem(cont)

- Example: Let $\mathbf{F}=\left\langle -y^2,x,z^2\right\rangle$ and $C$ be an intersection between cylinder $x^2+y^2=1$ and $y+z=2$. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$.




$$\vec{r}(t)=\left\langle \cos t,\sin t,2-\sin t\right\rangle$$

$$\mathbf{F}(\vec{r}(t))\cdot d\mathbf{r}=\left\langle -\sin^2 t,\cos t,(2-\sin t)^2\right\rangle\cdot\left\langle -\sin t,\cos t,-\cos t\right\rangle$$

$$=\sin^3 t+\cos^2 t-(2-\sin t)^2\cos t$$

- Thus direct integration won't be nice; therefore we try Stokes. Let the surface $S$ be elliptical region on plane $y+z=2$.

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_{S:z=2-y}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\iint_{D:x^2+y^2=1}\underbrace{\left\langle 0,0,1+2y\right\rangle}_{\operatorname{curl}\mathbf{F}}\cdot\underbrace{\left\langle -g_x,-g_y,1\right\rangle}_{\mathbf{n}}dA=\iint_D 1+2y\,dA$$

$$=\int_0^{2\pi}\int_0^1(1+2r\sin\theta)r\,dr\,d\theta=\int_0^{2\pi}\left(\frac{r^2}{2}+2\cdot\frac{r^3}{3}\sin\theta\right)\Bigg|_0^1 d\theta=\int_0^{2\pi}\frac{1}{2}+2\cdot\frac{1}{3}\sin\theta\,d\theta=\frac{1}{2}\theta-\frac{2}{3}\cos\theta\Bigg|_0^{2\pi}=\pi$$

# Divergence Theorem

- Let *E* be a simple solid region and let *S* be the boundary surface of *E*, given with positive (outward) orientation. Let **F** be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

- Note the similarity with Normal Component Green's Theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F}\left(x, y\right) dA$$

# Divergence Theorem(cont)

- **Proof**: Let **F**=<*P,Q,R*>, we want to show

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E P_x dV + \iiint_E Q_y dV + \iiint_E R_z dV \underset{show}{=} \oiint_S P\mathbf{i}\cdot\mathbf{n}dS + \oiint_S Q\mathbf{j}\cdot\mathbf{n}dS + \oiint_S R\mathbf{k}\cdot\mathbf{n}dS$$

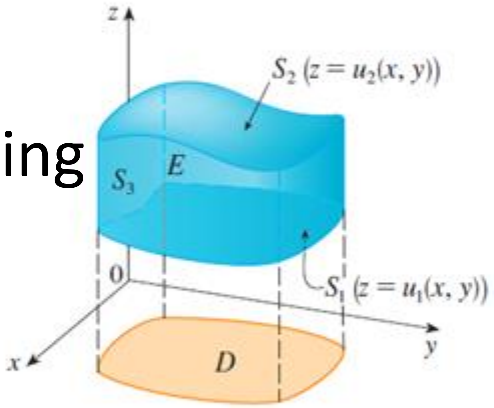
$$= \oiint_S \mathbf{F}\cdot\mathbf{n}dS = \oiint_S \mathbf{F}\cdot d\mathbf{S}$$

- Consider $E \underset{\text{type I}}{=} \{(x,y,z):(x,y)\in D, u_1(x,y)\le z\le u_2(x,y)\}$, then

$$\iiint_E R_z dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} R\,dz\,dA = \iint_D R(x,y,u_2(x,y)) - R(x,y,u_1(x,y))\,dA$$

$$= \oiint_{S_1} R\mathbf{k}\cdot\mathbf{n}dS + \cancel{\oiint_{S_3} R\mathbf{k}\cdot\mathbf{n}dS}^{\text{either } \mathbf{k}\cdot\mathbf{n}=0 \text{ or } S_3=\phi} - \oiint_{S_2} R\mathbf{k}\cdot\mathbf{n}dS = \oiint_S R\mathbf{k}\cdot\mathbf{n}dS$$

- $\iiint_E P_x dV = \oiint_S P\mathbf{i}\cdot\mathbf{n}dS$, $\iiint_E Q_y dV = \oiint_S Q\mathbf{j}\cdot\mathbf{n}dS$ are proved in a similar manner using the expressions for *E* as a type II or type III region, respectively.

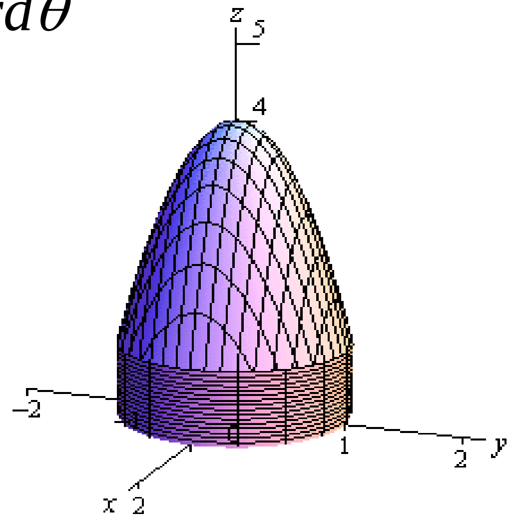


# Divergence Theorem(cont)

- **Example**: Let $\mathbf{F} = \left\langle xy, -\frac{1}{2}y^2, z \right\rangle$ and *S* be defined by $z = 4 - 3x^2 - 3y^2, 1 \le z \le 4$ on top, $x^2 + y^2 = 1, 0 \le z \le 1$ on sides and $z = 0$ at the bottom.

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E \cancel{y - y} + 1 dV = \int_0^{2\pi} \int_0^1 \int_0^{4-3r^2} r dz dr d\theta = \int_0^{2\pi} \int_0^1 rz \Big|_0^{4-3r^2} dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 r\left(4 - 3r^2\right) dr d\theta = 2\pi \left( 2r^2 - \frac{3}{4} r^4 \right)_0^1 = \frac{5}{2}\pi$$



# Divergence Theorem(cont)

- **Example**: Let $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$, *S* spherical solid of radius 2 in first octant.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV = 3\iiint_E x^2 + y^2 + z^2 dV = 3\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} \rho^4 \sin\varphi d\rho d\varphi d\theta =$$

$$= 3\int_0^{\pi/2}\int_0^{\pi/2} \left.\frac{\rho^5}{5}\right|_0^2 \sin\varphi d\varphi d\theta = \frac{3\cdot 32}{5}\frac{\pi}{2}\int_0^{\pi/2} \sin\varphi d\varphi = -\frac{48}{5}\cos\varphi\Big|_0^{\pi/2} = \frac{48}{5}$$

- **Example**: Let $\mathbf{F} = \langle 3y\cos z, x^2 e^z, x\sin y \rangle$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E 0 dV = 0$$

# Decision Tree