Vector Calculus

part I

By Dr. Michael Medvinsky, NCSU

online lectures 03/2020

Vector Field (definition)

- **Definition**: Vector Field is a function F that for each (x,y)\(x,y,z) assign a 2\3-dimensional vector, respectively:

$$\mathbf{F}(x,y)=\left\langle P(x,y),Q(x,y)\right\rangle$$

$$\mathbf{F}(x,y,z)=\left\langle P(x,y,z),Q(x,y,z),R(x,y,z)\right\rangle$$

- Examples of VF: gradient, direction field of differential equation.
- Vector field vs other functions we learned:

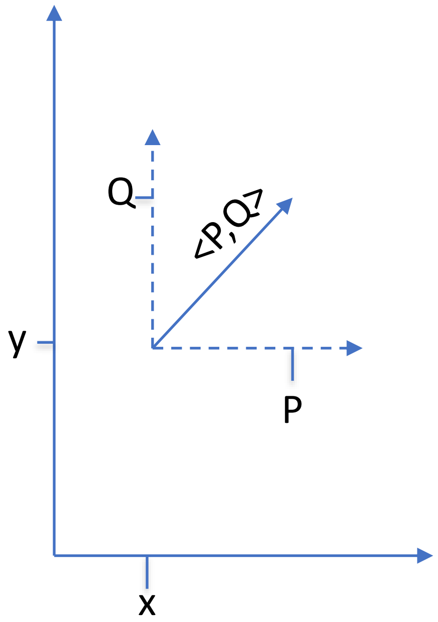
$f:\mathbb{R}^n\to\mathbb{R}$	function of $n=1,2,3$ variables
$\vec{r}:\mathbb{R}\to\mathbb{R}^n$	vector (of size $n=1,2,3$) valued function, e.g. parametric curve
$\vec{r}:\mathbb{R}^2\to\mathbb{R}^3$	parametric surface
$\mathbf{F}:\mathbb{R}^n\to\mathbb{R}^n$	vector field ($n=2,3$)

Vector Field (how to sketch it)

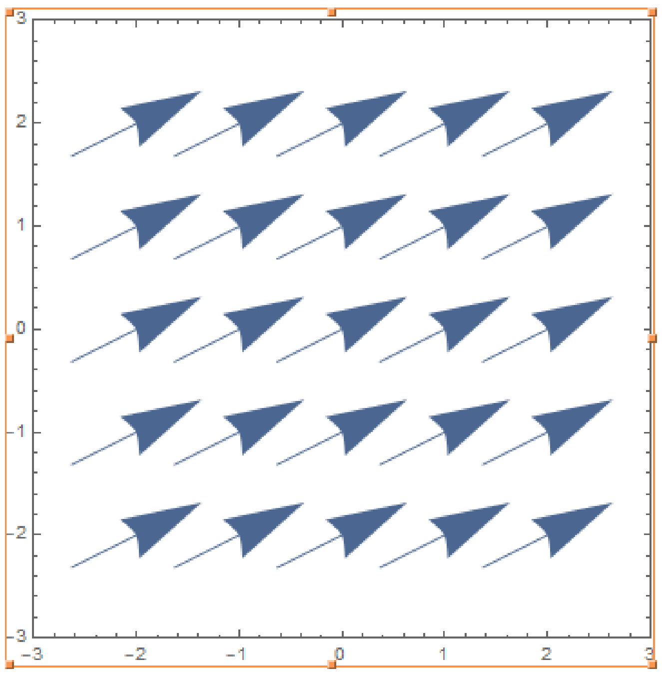
- We draw VF as vectors $\langle P(x,y),Q(x,y)\rangle \backslash \langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$ starting at points $(x,y)\backslash(x,y,z)$

Examples:

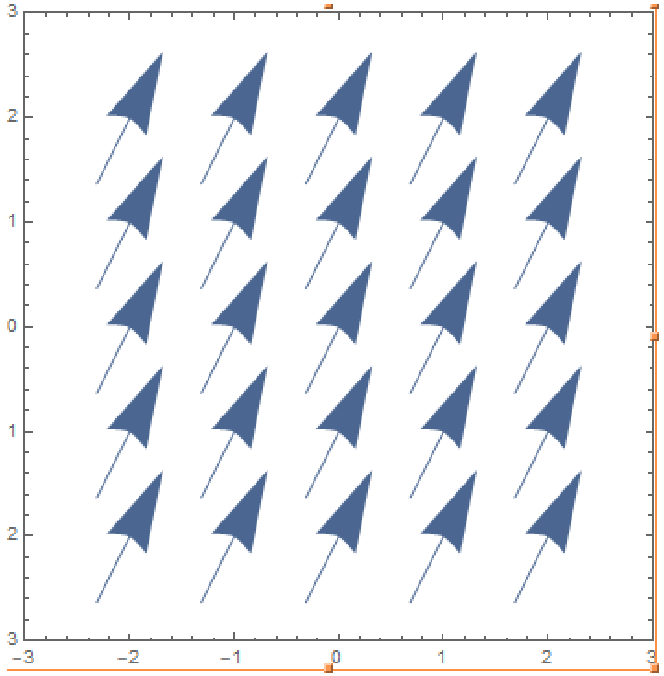
General case



Uniform\Constant VF: F=<2,1>



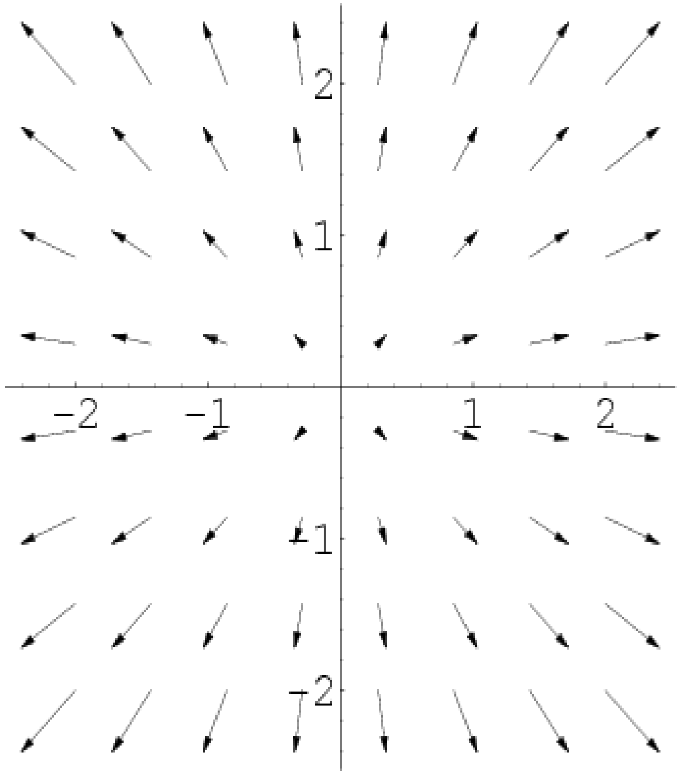
Uniform\Constant VF:F=<1,2>



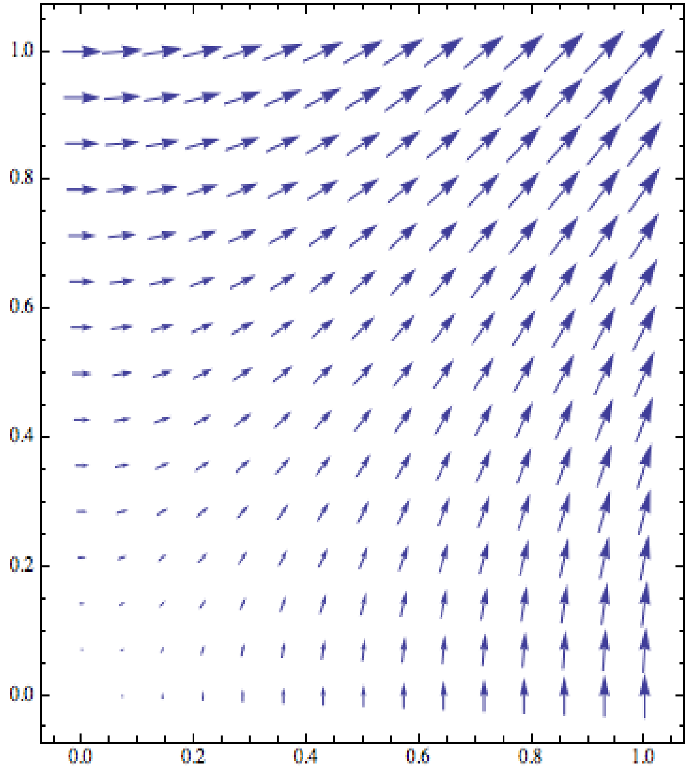
Vector Field (how to sketch it)

- More examples:

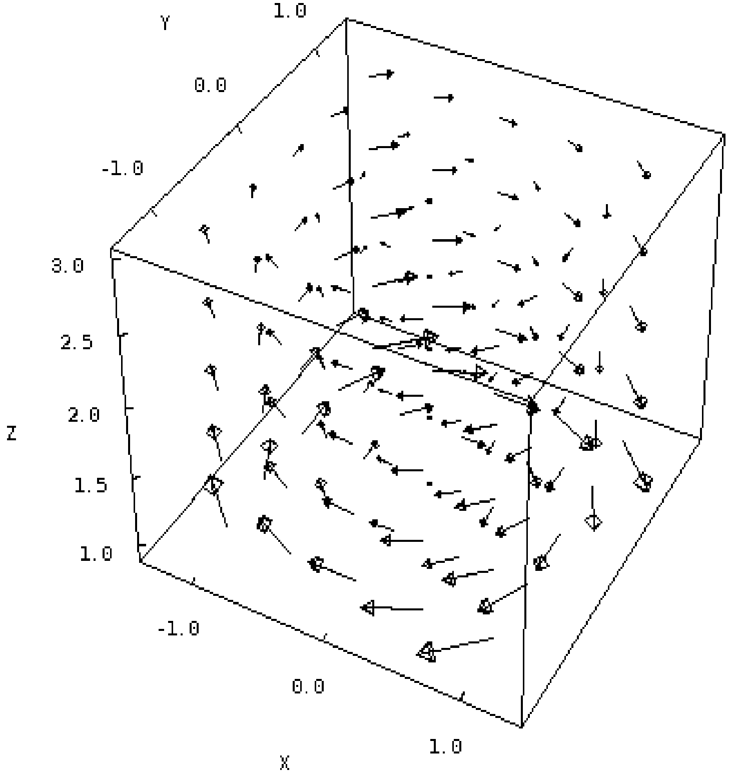
$$\mathbf{F}(x,y)=\langle x,y\rangle$$



$$f(x,y)=xy,\mathbf{F}=\vec{\nabla}f=\langle y,x\rangle$$

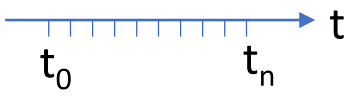


$$\mathbf{F}(x,y,z)=\left\langle \frac{y}{z},-\frac{x}{z},0\right\rangle$$



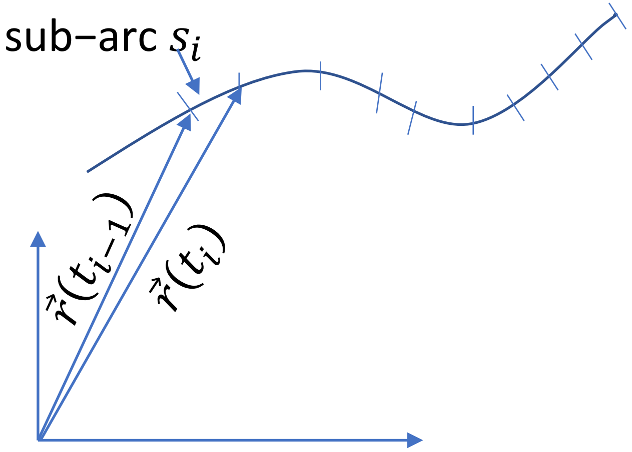
Line Integral (the idea)

- Consider smooth curve *C* be given by: $\vec{r}(t) = \langle x(t), y(t) \rangle, t \in [a,b]$
  recall: smooth means $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t) \neq 0$
- Divide [a,b] into subintervals $t_i = a + i\Delta t, \Delta t = \dfrac{b-a}{n}$



- Denote s_i a piece of *C* corresponding to $[t_{i-1}, t_i]$ and displacement as Δs_i .
- Denote $\langle x_i^*, y_i^* \rangle$ a sample point on s_i.
- Consider that function *f(x,y)* is defined along *C.*
- What do you think about the following *Rieman-like* sum

$$\sum_{i=1}^{n} f\left(x_i^*, y_i^*\right) \Delta s_i$$



Line Integral (definition)

- Let *f* be a function defined along a curve C given by

$$\vec{r}(t) = \langle x(t), y(t) \rangle \quad or \quad \vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad t \in [a,b]$$

then the line\contour\path\curve integral is defined by

$$\oint_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \quad or \quad \oint_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i$$

- *Recall: Arclength formula:*

$$L = \int_a^b \sqrt{x_t^2 + y_t^2}\,dt \quad or \quad L = \int_a^b \sqrt{x_t^2 + y_t^2 + z_t^2}\,dt \quad or \quad L = \int_a^b \left|\vec{r}\,'(t)\right| dt$$

Line Integral (2 theorems)

- 1) Let *f* be continuous function along curve *C* (defined as before):

$$\oint_C f(x,y)\,ds = \int_a^b f\left(x(t),y(t)\right)\frac{ds}{dt}dt = \int_a^b f\left(x(t),y(t)\right)\sqrt{x_t^2+y_t^2}\,dt \qquad \text{or}$$

$$\oint_C f(x,y,z)\,ds = \int_a^b f\left(x(t),y(t),z(t)\right)\frac{ds}{dt}dt = \int_a^b f\left(x(t),y(t)\right)\sqrt{x_t^2+y_t^2+z_t^2}\,dt \qquad \text{or}$$

$$\oint_C f(\vec{x})\,ds = \int_a^b f\left(\vec{r}(t)\right)\left|\vec{r}\,'(t)\right|dt,\ \text{for either } \vec{x}=(x,y) \text{ or } \vec{x}=(x,y,z)$$

  regardless of the parameterization as long as the curve traversed exactly once between a and b.

- 2) *Let* $C = C_1 \cup C_2 \cup \ldots$ *be piecewise smooth curve, then* $\oint_C f\,ds = \oint_{C_1} f\,ds + \oint_{C_2} f\,ds + \ldots$

Line Integral(examples)

- 1) Let $\vec{r}\left(t\right)=\left\langle 2\cos t,2\sin t\right\rangle, 0\le t\le \pi$ (half circle, r=2), evaluate $\oint_C x^2yds$

  Solution:
  We have $f\left(x,y\right)=x^2y\Rightarrow f\left(\vec{r}\left(t\right)\right)=f\left(2\cos t,2\sin t\right)=4\cos^2 t\cdot 2\sin t=8\cos^2 t\cdot \sin t$
  the arclength is $\left|\vec{r}\,'\left(t\right)\right|=2\left|\left\langle -\sin t,\cos t\right\rangle\right|=2$
  thus

$$\oint_C x^2yds=\int_0^\pi \underbrace{8\cos^2 t\cdot \sin t}_{f\left(\vec{r}\left(t\right)\right)}\cdot \underbrace{2}_{\left|\vec{r}\,'\left(t\right)\right|}dt=16\int_0^\pi \cos^2 t\cdot \sin t dt \underset{u=\cos t}{=} 16\int_{-1}^{1} u^2du=\frac{32}{3}$$

Line Integral(examples)

- 2) Let $C_1 = r_1(t) = \left\langle \sqrt{8}t, 4t, 5t \right\rangle, 0 \le t \le 1$ and $C_2 = r_2(t) = \langle 1, 2, -5t \rangle, -1 \le t \le 0$ evaluate $\oint_{C_1 \cup C_2} x + y + zds$

Solution:

$$\oint_{C_1} x + y + zds = \int_0^1 \underbrace{\left(\sqrt{8}t + 4t + 5t\right)}_{f(\vec{r}(t))} \cdot \underbrace{\sqrt{8 + 16 + 25}}_{|\vec{r}\,'(t)|} dt = 7\left(9 + \sqrt{8}\right)\int_0^1 tdt = 7\left(9 + \sqrt{8}\right)\frac{t^2}{2}\Big|_0^1 = \frac{7\left(9 + \sqrt{8}\right)}{2}$$

$$\oint_{C_2} x + y + zds = \int_{-1}^0 \underbrace{(1 + 2 - 5t)}_{f(\vec{r}(t))} \cdot \underbrace{\sqrt{0 + 0 + 25}}_{|\vec{r}\,'(t)|} dt = 5\int_{-1}^0 3 - 5tdt = 5\left(3t - 5 \cdot \frac{t^2}{2}\right)_{-1}^0 = -5\left(-3 - 5 \cdot \frac{1}{2}\right) = 15 + \frac{25}{2} = \frac{55}{2}$$

$$\oint_{C_1 \cup C_2} x + y + zds = \oint_{C_1} x + y + zds + \oint_{C_2} x + y + zds = \frac{7\left(9 + \sqrt{8}\right)}{2} + \frac{55}{2} = 59 + \frac{7\sqrt{8}}{2}$$

Line Integral(examples)

- 3) Evaluate $\oint_C \frac{xy}{\sqrt{13}}\,ds$ where *C* is a line between (1,2) and (3,-1)

Solution: parameterize $\vec{r}(t) = (1-t)\langle 1,2\rangle + t\langle 3,-1\rangle = \langle 1+2t, 2-3t\rangle$

to get $\oint_C \frac{xy}{\sqrt{13}}\,ds = \int_0^1 \underbrace{\frac{(1+2t)(2-3t)}{\cancel{\sqrt{13}}}}_{f(\vec{r}(t))} \cdot \underbrace{\cancel{\sqrt{4+9}}}_{|\vec{r}\,'(t)|}\,dt = \int_0^1 2+t-6t^2 dt = \frac{1}{2}$

Alternative Solution: parameterize $\vec{r}(t) = (1-t)\langle 3,-1\rangle + t\langle 1,2\rangle = \langle 3-2t, -1+3t\rangle$

to get $\oint_C \frac{xy}{\sqrt{13}}\,ds = \int_0^1 \underbrace{\frac{(3-2t)(-1+3t)}{\cancel{\sqrt{13}}}}_{f(\vec{r}(t))} \cdot \underbrace{\cancel{\sqrt{4+9}}}_{|\vec{r}\,'(t)|}\,dt = \int_0^1 -3+11t-6t^2 dt = \frac{1}{2}$

Line Integral(theorem + definition)

- In previous example we traversed a curve (line) in two opposite direction and got the same result – it didn't happen by an accident.
- **Theorem**: Denote by *-C* the same curve as *C*, but with different direction: $\oint_C f\,ds = \oint_{-C} f\,ds$
- **Definition**: Denote $\oint_C f\,ds$ a line integral with respect to arclength, a line integral with respect to x: $\oint_C f(x,y)\,dx = \int_a^b f\left(x(t),y(t)\right)x'(t)\,dt$, with respect to y: $\oint_C f(x,y)\,dy = \int_a^b f\left(x(t),y(t)\right)y'(t)\,dt$ and analogically in 3D $\oint_C f(x,y,z)\,dx, \oint_C f(x,y,z)\,dy, \oint_C f(x,y,z)\,dz$.
  They often occur together, e.g.

$$\oint_C f(x,y)\,dx + \oint_C g(x,y)\,dy = \oint_C f(x,y)\,dx + g(x,y)\,dy$$

Line Integral(example)

- Evaluate $\oint\limits_{\substack{\langle 2+t,4t,5t\rangle,\\ 0\le t\le 1}} ydx+zdy+xdz$

Solution:

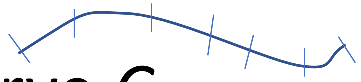
$$\oint\limits_{\substack{\langle 2+t,4t,5t\rangle,\\ 0\le t\le 1}} ydx+zdy+xdz=\int_0^1 \underset{y}{4t}\underbrace{\frac{d}{dt}\left(2+t\right)}_{x'(t)}dt+\int_0^1 \underset{z}{5t}\underbrace{\frac{d}{dt}\left(4t\right)}_{y'(t)}dt+\int_0^1 \underbrace{\left(2+t\right)}_{x}\underbrace{\frac{d}{dt}\left(5t\right)}_{z'(t)}dt$$

$$=\int_0^1 4t+20t+5\left(2+t\right)dt=\int_0^1 29t+10dt=24.5$$

Line Integral of Vector Field

- **Reminder**:
  - A work done by variable force *f(x)* in moving a particle from *a* to *b* along the *x*-axis is given by $W = \int_a^b f(x)\,dx$.
  - A work done by a constant force **F** in moving object from point P to point Q in space is $W = \mathbf{F} \cdot \overrightarrow{PQ}$.
  - Unit tangent vector: $\mathbf{T}(t) = \dfrac{\vec{r}\,'(t)}{\left|\vec{r}\,'(t)\right|}$

$\mathbf{F}$ is $\sim constant\ on\ s_i$



- Consider now a variable force **F**(*x,y,z*) along a smooth curve *C*.
  - Divide *C* into number of a small enough sub-arcs so that the force is roughly constant on each sub-arc.
  - The displacement vector becomes unit tangent (**T**) times displacement (Δs_i):

$$\overrightarrow{PQ} = \Delta s_j \mathbf{T}\left(x\left(t_i^*\right), y\left(t_i^*\right), z\left(t_i^*\right)\right), t_i^* \in \left[t_{i-1}, t_i\right]$$

Line Integral of Vector Field(cont)

- Finally the work of **F**(*x,y,z*) along *C* is given by

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}\mathbf{F}\left(x\left(t_i^*\right), y\left(t_i^*\right), z\left(t_i^*\right)\right)\cdot\mathbf{T}\left(x\left(t_i^*\right), y\left(t_i^*\right), z\left(t_i^*\right)\right)\Delta s_j$$

$$= \oint_C \mathbf{F}\left(x, y, z\right)\cdot\mathbf{T}\left(x, y, z\right)ds = \oint_C \mathbf{F}\cdot\mathbf{T}ds$$

- Denote $d\mathbf{r} = \vec{r}\,'\left(t\right)$ or $d\vec{r} = \vec{r}\,'\left(t\right)$

$$\oint_C \mathbf{F}\cdot\mathbf{T}ds = \int_a^b\left(\mathbf{F}\left(\vec{r}\left(t\right)\right)\cdot\frac{\vec{r}\,'\left(t\right)}{\cancel{\left|\vec{r}\,'\left(t\right)\right|}}\right)\cancel{\left|\vec{r}\,'\left(t\right)\right|}dt = \int_a^b\mathbf{F}\left(\vec{r}\left(t\right)\right)\cdot\vec{r}\,'\left(t\right)dt \equiv \int_a^b \underbrace{\mathbf{F}}_{\mathbf{F}\left(\vec{r}\left(t\right)\right)}\cdot\underbrace{d\mathbf{r}}_{d\vec{r}\,'}$$

Line Integral of Vector(example)

- Let VF be given by $\mathbf{F}=\langle x, x+y, x+y+z\rangle$ and
- the curve C by $\vec{r}(t)=\langle \sin t, \cos t, \sin t+\cos t\rangle, 0\le t\le 2\pi$
  Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$.

Solution:

$$\mathbf{F}\cdot d\mathbf{r} = \underbrace{\langle \sin t, \cos t+\sin t, 2(\sin t+\cos t)\rangle}_{\mathbf{F}(\vec{r}(t))}\cdot\underbrace{\langle \cos t, -\sin t, \cos t-\sin t\rangle dt}_{=r'(t)dt}$$

$$=\left(2\cos^2 t-3\sin^2 t\right)dt=\left(\underbrace{1+\cos 2t}_{=2\cos^2 t}-\frac{3}{2}\underbrace{(1-\cos 2t)}_{=2\sin^2 t}\right)dt$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi} 1+\cos 2t-\frac{3}{2}(1-\cos 2t)\,dt=\left(t+\frac{\sin 2t}{2}-\frac{3}{2}\left(t-\frac{\sin 2t}{2}\right)\right)_0^{2\pi}=-\pi$$

Line integral of Vector vs Scalar fields

- Let $\mathbf{F}(x, y, z) = \left\langle P\left(x, y, z\right), Q\left(x, y, z\right), R\left(x, y, z\right)\right\rangle$
  and $\vec{r}\left(t\right) = \left\langle x\left(t\right), y\left(t\right), z\left(t\right)\right\rangle, a \le t \le b$
  then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}\left(\vec{r}\left(t\right)\right) \cdot \overline{r}\,'\left(t\right) dt$$

$$= \int_a^b \left\langle P\left(x, y, z\right), Q\left(x, y, z\right), R\left(x, y, z\right)\right\rangle \cdot \left\langle x'\left(t\right), y'\left(t\right), z'\left(t\right)\right\rangle dt$$

$$= \int_a^b P\left(x, y, z\right) x'\left(t\right) + Q\left(x, y, z\right) y'\left(t\right) + R\left(x, y, z\right) z'\left(t\right) dt$$

$$= \int_a^b P\left(x, y, z\right) dx + Q\left(x, y, z\right) dy + R\left(x, y, z\right) dz$$

Fundamental Theorem for Line Integrals

- **Recall:** Fundamental Theorem of Calculus (FTC) $\int_a^b F'(x)\,dx = F(b) - F(a)$
- Definition: A vector field **F** is called a **conservative vector field** if there exist a **potential**, a function *f*, such that $\mathbf{F} = \vec{\nabla} f$.
- **Theorem**: Let *C* be a smooth curve given by $\vec{r}(t), a \le t \le b$. Let **F** be a *continuous conservative vector field*, and *f* is a differentiable function satisfying $\mathbf{F} = \vec{\nabla} f$. Then, .

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \nabla f \cdot d\mathbf{r} = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right)$$

Fundamental Theorem for Line Integrals(cont)

- Proof:

$$\oint_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f\left(\vec{r}(t)\right)\cdot \vec{r}\,'(t)\,dt = \int_a^b \left\langle f_x, f_y, f_z \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

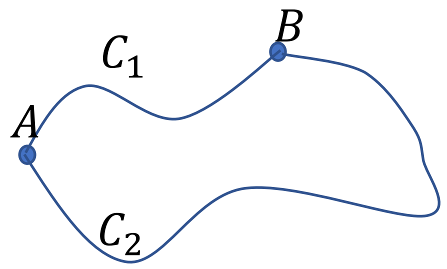
$$= \int_a^b \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \right) dt = \int_a^b \frac{d}{dt} f\left(\vec{r}(t)\right) dt$$

$$= f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right)$$

Fundamental Theorem for Line Integrals(cont)

- **Definition**: Let **F** be continuous on domain D. The integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is called **independent of path in D** if for *any two* curves C_1, C_2 with the same initial and end points, we have

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r}$$



- **Corollary**: A line integral of a conservative vector field is independent of path.
- **Definition**: A curve *C* is called closed if its terminal points coincides.

Fundamental Theorem for Line Integrals(cont)

- **Theorem**: The integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D **if and only if** $\oint_C \mathbf{F}\cdot d\mathbf{r}=0$ on any closed curve *C*.

Proof: (➔) Let $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D. Let *C* be *arbitrary* closed curve. Choose any two points on *C*, A and B. Let C_1 be the curve from A to B, and C_2 from B to A, so that $C=C_1\cup C_2$, then

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_{C_1}\mathbf{F}\cdot d\mathbf{r}+\oint_{C_2}\mathbf{F}\cdot d\mathbf{r}=\oint_{C_1}\mathbf{F}\cdot d\mathbf{r}-\oint_{-C_2}\mathbf{F}\cdot d\mathbf{r}=0$$

(🡐) Let $\oint_C \mathbf{F}\cdot d\mathbf{r}=0$ on any closed curve *C* in D. Choose A,B∈D and let C_1, C_2 be arbitrary paths from A to B. $C=-C_1\cup C_2$ is closed curve, thus

$$0=\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_{-C_1}\mathbf{F}\cdot d\mathbf{r}+\oint_{C_2}\mathbf{F}\cdot d\mathbf{r}=-\oint_{C_1}\mathbf{F}\cdot d\mathbf{r}+\oint_{C_2}\mathbf{F}\cdot d\mathbf{r}\Rightarrow\oint_{C_1}\mathbf{F}\cdot d\mathbf{r}=\oint_{C_2}\mathbf{F}\cdot d\mathbf{r}$$

Fundamental Theorem for Line Integrals(cont)

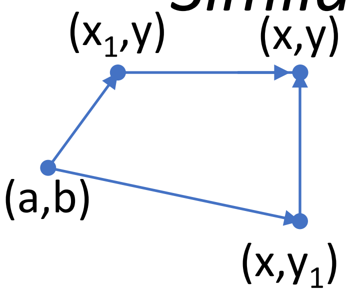
- **Theorem**: Suppose **F**=<P,Q> is continuous vector field on an open connected region D. If $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D, then **F** is conservative vector field in D, that is there is *f* such that $\mathbf{F}=\vec{\nabla} f$.

Proof: Let (*a,b*)∈D be arbitrary fixed point. Define $f(x,y)=\int_{(a,b)}^{(x,y)}\mathbf{F}\cdot d\mathbf{r}$.

Due to independency of path we can choose path *C* from (*a*,*b*) to (*x*,*y*) that crosses (x_1,*y*)∈D, x_1 is const.

$$f_x(x,y)=\frac{d}{dx}\int_{(a,b)}^{(x,y)}\mathbf{F}\cdot d\mathbf{r}=\cancel{\frac{d}{dx}\int_{(a,b)}^{(x_1,y)}\mathbf{F}\cdot d\mathbf{r}}^{=0,\text{ no }x}+\frac{d}{dx}\int_{(x_1,y)}^{(x,y)}\mathbf{F}\cdot d\mathbf{r}=\frac{d}{dx}\int_{(x_1,y)}^{(x,y)}Pdx+\overset{dy=0}{\cancel{Qdy}}=\frac{d}{dx}\int_{x_1}^{x}Pdx\underset{FTC}{=}P$$

Similarly,



$$f_y(x,y)=\frac{d}{dy}\int_{(a,b)}^{(x,y)}\mathbf{F}\cdot d\mathbf{r}=\frac{d}{dy}\int_{(a,b)}^{(x,y_1)}\mathbf{F}\cdot d\mathbf{r}+\cancel{\frac{d}{dy}\int_{(x,y_1)}^{(x,y)}\mathbf{F}\cdot d\mathbf{r}}^{=0,\text{ no y}}=\frac{d}{dy}\int_{(x,y_1)}^{(x,y)}\overset{dx=0}{\cancel{Pdx}}+Qdy=\frac{d}{dy}\int_{y_1}^{y}Qdy\underset{FTC}{=}Q$$