# Vector Calculus

## part II

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# Fundamental Theorem for Line Integrals(cont)

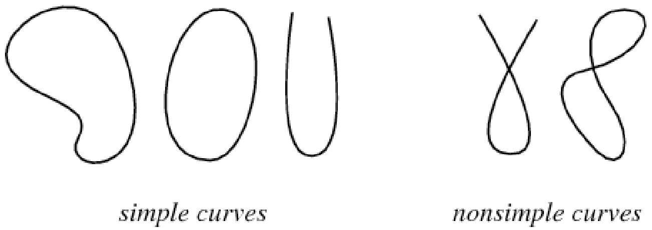
- **Theorem**: Suppose **F**=<P,Q> is a conservative vector field and P,Q has continuous first order partial derivatives on domain D, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

**Proof**: Let *f* be the potential, i.e. $\langle P, Q \rangle = \mathbf{F} = \vec{\nabla} f = \langle f_x, f_y \rangle$, therefore

$$f_{xy} = \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = f_{yx}$$

# Fundamental Theorem for Line Integrals(cont)

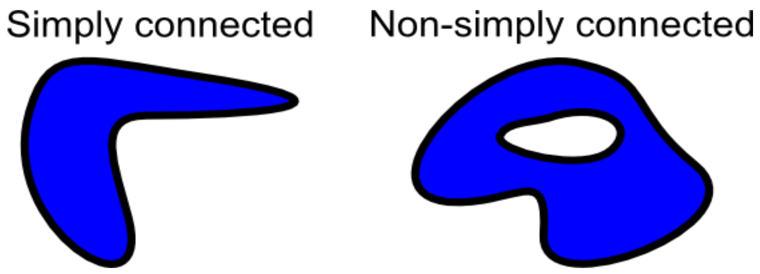



- **Definitions**:

1) A **simply connected curve** is a curve that doesn't intersect itself between endpoints.

2) A **simple closed curve** is a curve with $\vec{r}(a)=\vec{r}(b)$ but $\vec{r}(t_1)\neq\vec{r}(t_2)$ for any $a<t_1<t_2<b$.

3) A **simply connected region**: is a region D in which every simple closed curve encloses only points from D. In other words D consist of one piece and has no hole.



# Fundamental Theorem for Line Integrals(cont)

- **Theorem**: Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field on an open simply connected region D. If P,Q have continuous first order partial derivatives on domain D and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then **F** is conservative.

- **Example**: Determine whether $\mathbf{F}(x, y) = \langle x \sin y, y \sin x \rangle$ is conservative.
  **Solution**: Not conservative, since

$$P_y = (x \sin y)_y = x \cos y \neq y \cos x = (y \sin x)_x = Q_x$$

# Fundamental Theorem for Line Integrals(cont)

- **Example**: Show that $\mathbf{F}(x,y)=\langle P,Q\rangle=\langle x+y,x-y\rangle$ and find the potential.

**Solution**: $P_y=(x+y)_y=1=(x-y)_x=Q_x$ , indeed **F** is conservative.

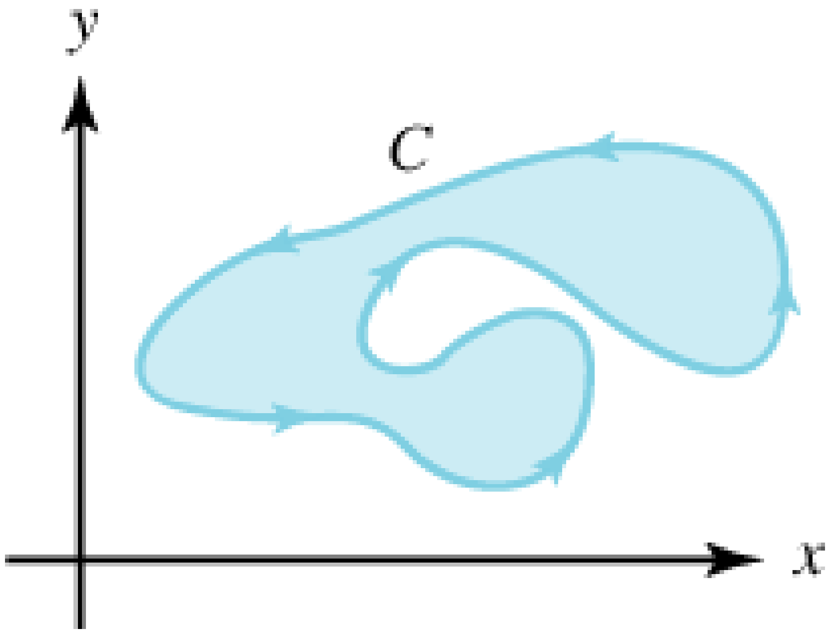
- To find the potential start with
$$f(x,y)=\int f_x(x,y)\,dx=\int x+y\,dx=\frac{x^2}{2}+yx+g(y)$$
note that the constant of integration can be function of y.
- To find *g* differentiate and compare to *Q*: $f_y=x+g'(y)=x-y$
*to get* $g(y)=\int g'(y)\,dx=-\int y\,dx=-\frac{y^2}{2}+const$
- Finally, since any potential works, set *const=0 to get*
$$f(x,y)=\frac{x^2}{2}+yx-\frac{y^2}{2}$$

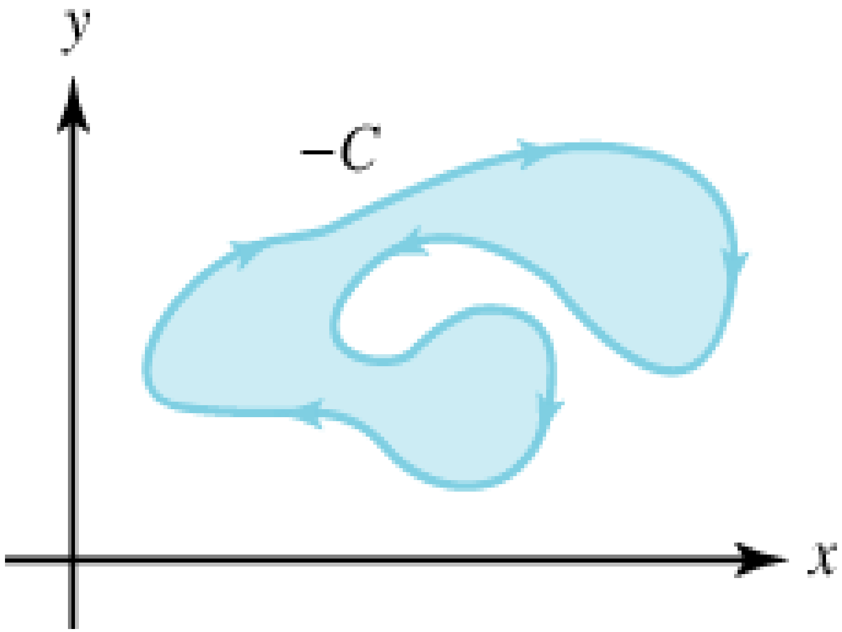
# Green’s Theorem

- **Definition**: A simple closed curve is said to be **positive oriented** if it traversed **counterclockwise**.

Counterclockwise – positively oriented

Clockwise – negatively oriented



# Green’s Theorem(the theorem)

- **Green’s Theorem**: Let *C* be **positively oriented piecewise-smooth, simple closed curve** in the plane and let ***D*** be the **region bounded by *C***. If *P* and *Q* have continuous partial derivatives on an open region that contains *D*, then

$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

- **Note**: The circle on the line integral ($\oint$) is sometime related to the positive oriented curve and sometime even drawn with an arrow on the circle: $\oint$

# Green’s Theorem(cont)

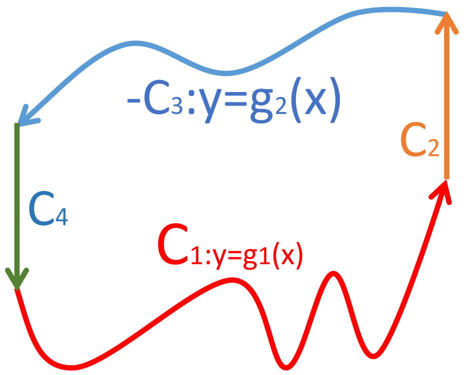
- One views the Green’s theorem as a counterpart of Fundamental Theorem of Calculus
  - Green’s theorem $$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \oint_{\partial D} P dx + Q dy$$
  - FTC theorem $$\int_a^b F'(x)\, dx = F(b) - F(a)$$
- Notice that in both, the left side is on the domain while the right one is at the boundary of the domain.

# Green's Theorem(proof)

**Proof**:

- Formulate *D* as domain of type I and show that $\oint_{\partial D} P dx = -\iint_D \frac{\partial P}{\partial y} dA$

  thus, let $D = \{a \le x \le b, g_1(x) \le y \le g_2(x)\}$
  and let $C = C_1 \cup C_2 \cup C_3 \cup C_4$, as depicted




$$\oint_{\partial D} P dx = \oint_{\langle x, g_1(x)\rangle} P dx + \oint_{\langle b, g_1(b)\rangle(1-t)+\langle b, g_2(b)\rangle t} P dx - \oint_{\langle x, g_2(x)\rangle} P dx + \oint_{\langle a, g_2(a)\rangle(1-t)+\langle a, g_1(a)\rangle t} P dx$$

$$= \int_a^b P(x, g_1(x)) dx + \cancel{\int_b^b P dx} - \int_a^b P(x, g_2(x)) dx + \cancel{\int_a^a P dx}$$

which is the same as $-\iint_D \frac{\partial P}{\partial y} dA = -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} dy dx = \int_a^b \int_{g_2(x)}^{g_1(x)} \frac{\partial P}{\partial y} dy dx = \int_a^b P(x, g_1(x)) - P(x, g_2(x)) dx$

- Similarly, one formulate *D* as domain of type II to show that $\oint_{\partial D} Q dy = \iint_D \frac{\partial Q}{\partial x} dA$

# Green's Theorem(cont)

- Example: Let D be square $[0,2]\times[0,2]$. Evaluate $\oint_{\partial D}\left(x^2-xy^3\right)dx+\left(y^2-2xy\right)dy$
  **Solution**: Using Green's theorem,

$$\oint_{\partial D}\underbrace{\left(x^2-xy^3\right)}_{P}dx+\underbrace{\left(y^2-2xy\right)}_{Q}dy=\iint_D\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}dA=\iint_D\frac{\partial}{\partial x}\left(y^2-2xy\right)-\frac{\partial}{\partial y}\left(x^2-xy^3\right)dA$$

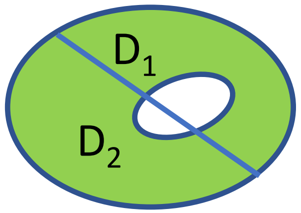
$$=\int_0^2\int_0^2-2y+3xy^2dxdy=\int_0^2\left(-2xy+3\frac{x^2}{2}y^2\right)_0^2dy=\int_0^2\left(-4y+3\frac{2^{\cancel{2}}}{\cancel{2}}y^2\right)_0^2dy=\left(-2y^2+2y^3\right)_0^2=8$$

- Verify $\oint_{\partial D}\left(x^2-xy^3\right)dx+\left(y^2-2xy\right)dy=\int_{\langle x,0\rangle}+\int_{\langle 2,y\rangle}-\int_{\langle x,2\rangle}-\int_{\langle 0,y\rangle}=$

$$=\int_0^2x^2dx+\int_0^2y^2-4ydy-\int_0^2x^2-2^3xdx-\int_0^2y^2dy=8\int_0^2xdx-4\int_0^2ydy=4x^2\Big|_0^2-2y^2\Big|_0^2=16-8=8$$

# Green's Theorem(extensions)

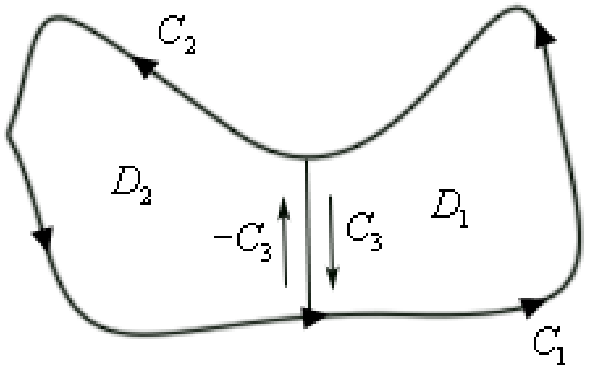
- How to use Green's theorem beyond its original formulation?
  - In the case when the curve $C$ is not closed (but its line integral isn't "nice"):
    - Connect the endpoints of $C$ with any **simple curve** $C_1$ to get $C_2 = C \cup C_1$
    - Now, $\int_{C_2}$ can conveniently(?) be evaluated using Green's theorem and $\int_C = \int_{C_2} - \int_{C_1}$
    - Hint: The best choice of $C_1$ will make $\int_{C_1}$ easy.
- In the case the region D has a hole, i.e. is not a simply connected.
    - Rewrite D as union of simply connected regions (see example)
    - Use the version of Green's theorem for Union of Domains (TBD on next slide)



# Green’s Theorem(extensions)

- **Theorem**: Let *D* be a domain. Rewrite *D* as union of 2 subdomains, e.g. $D = D_1 \cup D_2$, let $\partial D = C_1 \cup C_2$ and $C_3 = D_1 \cap D_2$, such that $\partial D_1 = C_1 \cup C_3$ and $\partial D_2 = C_2 \cup (-C_3)$, then

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \oint_{C_1 \cup C_3} Pdx + Qdy + \oint_{C_2 \cup (-C_3)} Pdx + Qdy$$




$$= \oint_{C_1} Pdx + Qdy + \oint_{C_2} Pdx + Qdy = \oint_{C_1 \cup C_2} Pdx + Qdy$$

# Green's Theorem(extensions)

- **Example**: Evaluate $A = \iint_D dA$.
- **Solution**: For a smart use of Green's Theorem: choose any *P* and *Q*, such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.
  - For example $P = 0, Q = x$, gives

$$A = \iint_D dA = \oint_{\partial D} x dy$$

# Green's Theorem(extensions)

- Let C: $\vec{r}(t)=\left\langle t,\sqrt{t-t^2}\right\rangle, t\in[0,1]$. Evaluate: $\oint_C \underbrace{\left(e^x\sin y-y^2+x\right)}_{P}dx+\underbrace{\left(e^x\cos y-\cos y^2\right)}_{Q}dy$
- **Solution**: reformulate the curve as $y=\sqrt{x-x^2}$ or $y^2+x^2=x$ which is a half circle, or in polar coordinates $r=\cos\theta, 0\le\theta\le\pi/2$. Connect the ends of the half circle with a line along x-axis, from 0 to 1.

$$\oint_C Pdx+Qdy=\oint_{C\cup C_1} Pdx+Qdy-\oint_{C_1} Pdx+Qdy=\iint_R Q_x-P_y dA-\oint_{\langle x,0\rangle} Pdx+Qdy=\frac{1}{6}-\frac{1}{2}=-\frac{1}{3}$$

$$\iint_R Q_x-P_y dA=\iint_R e^x\cos y-\left(e^x\cos y-2y\right)dA=\iint_R 2y dA=\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} 2r\sin\theta\cdot r dr d\theta=\frac{1}{6}$$

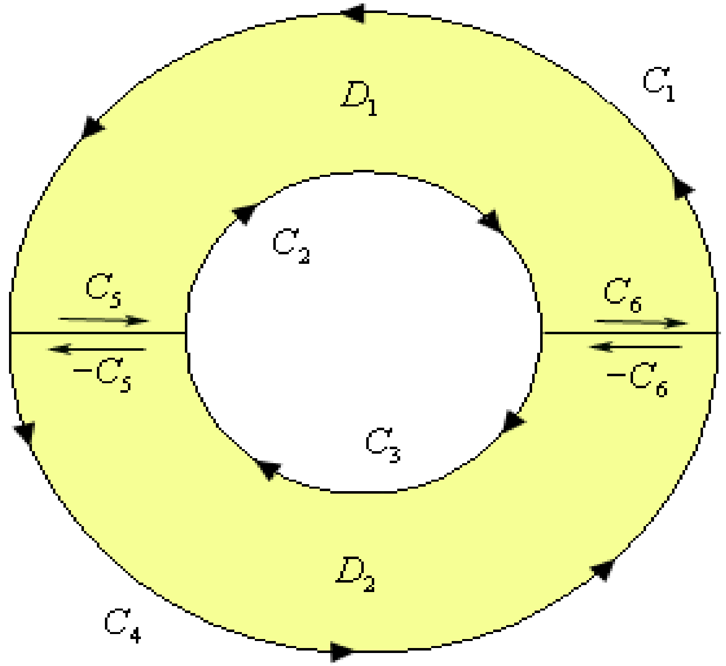
$$\oint_{\langle x,0\rangle} Pdx+\overbrace{Qdy}^{=0}=\int_0^1\left(e^t\sin 0-0^2+t\right)\frac{d}{dt}t dt=\left.\frac{t^2}{2}\right|_0^1=\frac{1}{2}$$

# Green's Theorem(extensions)

- **Example**: Let *C* be a ring with radiuses 1 and 2 centered at the origin.

$$\oint_C \underbrace{y^3}_{P}\, dx + \underbrace{x^3}_{Q}\, dy = \iint_{D_1} \underbrace{-3x^2}_{Q_x} \underbrace{- 3y^2}_{P_y}\, dA + \iint_{D_2} \underbrace{-3x^2}_{Q_x} \underbrace{- 3y^2}_{P_y}\, dA$$

$$= -3\int_0^{2\pi}\int_1^2 r^3 dr d\theta = -3\cdot 2\pi \left.\frac{r^4}{4}\right|_1^2 = -\frac{45}{2}\pi$$



# Curl and Divergence

- Let **F=**<*P,Q,R*> be a vector field on $\mathbb{R}^3$. Assume that all partial derivatives of *P*,*Q*,*R* exists, then
  - the curl of **F** is defined as

$$\operatorname{curl}\mathbf{F} = \vec{\nabla}\times\mathbf{F} = \det\begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

  - the divergence of **F** is defined as

$$\operatorname{div}\mathbf{F} = \vec{\nabla}\cdot\mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R\rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

# Curl and Divergence(cont)

- Example: Let $f(x,y,z) = x\sin yz$. Then $\mathbf{F} = \vec{\nabla} f = \langle \sin yz, xz\cos yz, xy\cos yz\rangle$,

$$\begin{aligned} \operatorname{div}\mathbf{F} &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle \sin yz, xz\cos yz, xy\cos yz\rangle \\ &= 0 + \left(-xz^2 \sin yz\right) + \left(-xy^2 \sin yz\right) = -x\left(y^2 + z^2\right)\sin yz \end{aligned}$$

and $\operatorname{curl}\mathbf{F} = 0$ since

$$\frac{\partial R}{\partial y} = x\cos yz - xyz\sin yz = \frac{\partial Q}{\partial z},$$

$$\frac{\partial P}{\partial z} = y\cos yz = \frac{\partial R}{\partial x}, \text{and}$$

$$\frac{\partial Q}{\partial x} = z\cos yz = \frac{\partial P}{\partial y}$$

# Curl and Divergence (cont)

- **Theorem**: Suppose f(x,y,z) has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = curl\left\langle f_x, f_y, f_z \right\rangle = \left\langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \right\rangle = 0$$

- **Theorem**: If **F** is vector field defined on $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl}\mathbf{F} = 0$, then **F** is a conservative vector field.

- **Theorem**: Suppose **F=<*P,Q,R*>** is a vector field on $\mathbb{R}^3$ and has continuous second-order partial derivatives, then

$$\operatorname{div}\operatorname{curl}\mathbf{F} = \operatorname{div}\left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle = \left( \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} \right) + \left( \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} \right) + \left( \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} \right) = 0$$