Vector Calculus

part III

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Curl and Divergence (cont)

- *Recall:* $\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial D} \mathbf{F} \cdot \mathbf{T} ds$, *where* $\mathbf{T}$ is the unit tangent vector.
- **Green's theorem in vector form (Tangential Component)**: Let $\mathbf{F} = \langle P(x,y), Q(x,y), 0\rangle$, *then*

$$\iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} dA = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

- *Proof:*

$$\operatorname{curl} \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, 0 \rangle = \left\langle 0 - \underbrace{\frac{\partial Q}{\partial z}}_{=0}, \underbrace{\frac{\partial P}{\partial z}}_{=0} - 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

$$\iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \underbrace{\mathbf{k} \cdot \mathbf{k}}_{=1} dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \oint_{\partial D} P dx + Q dy = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

Curl and Divergence (cont)

- **Recall: unit normal vector is given by** $\mathbf{n}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \underset{}{\overset{\text{in } 2D,\ \text{verify!}}{=}} \dfrac{\langle y'(t), -x'(t)\rangle}{|\vec{r}\,'(t)|}$
- **Green's theorem in vector form (Normal Component)**: Let **F**=<*P,Q*> be a vector field and $\vec{r}(t) = \langle x(t), y(t)\rangle$, then

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}(x,y)\,dA$$

- **Proof**:

$$\mathbf{F}(\vec{r}(t))\cdot\mathbf{n} = \langle P(x(t),y(t)), Q(x(t),y(t))\rangle\cdot\frac{\langle y'(t), -x'(t)\rangle}{|\vec{r}\,'(t)|} = \frac{Py'(t)-Qx'(t)}{|\vec{r}\,'(t)|}$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}ds = \int_a^b \frac{Py'(t)+Qd'(t)}{|\vec{r}\,'(t)|}|\vec{r}\,'(t)|dt = \int_a^b Pdy - Qdx$$

$$= \int_a^b (-Q)dx + Pdy = \iint_D \frac{\partial P}{\partial x} - \frac{\partial(-Q)}{\partial y}dA \underset{Green's}{=} \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}dA = \iint_D \operatorname{div}\mathbf{F}(x,y)dA$$

Curl and Divergence (cont)

- **Example**: Evaluate $\iint_D y^2 - x^2 dA$, where *D* is unit disk.
- **Solution**: Previously we would evaluate it directly using a trigonometric identity

$$\iint_D y^2 - x^2 dA = \int_0^{2\pi}\int_0^1 r^2\left(\sin^2\theta - \cos^2\theta\right) r dr d\theta = -\int_0^{2\pi}\int_0^1 r^3 \cos 2\theta dr d\theta = -\left(\frac{r^4}{4}\right)_0^1 \left(\frac{\sin 2\theta}{2}\right)_0^{2\pi} = -\frac{1}{4}\cdot 0 = 0$$

- Now we can do something else, let $\vec{r}(t) = \langle \cos t, \sin t \rangle$ and $\mathbf{F} = \langle x^2 y, xy^2, 0 \rangle$, then $y^2 - x^2 = (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}$, and

$$\iint_D y^2 - x^2 dA = \oint_{\partial D} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \underbrace{\langle \cos^2 t \sin t, \cos t \sin^2 \theta, 0 \rangle}_{\mathbf{F}(\vec{r}(t))} \cdot \underbrace{\langle \sin t, -\cos t, 0 \rangle}_{\vec{r}\,'(t)} dt = \int_0^{2\pi} \cos^2 t \sin^2 t - \cos^2 t \sin^2 t dt = 0$$

- Furthermore, notice that **F** is conservative VF (since $\mathbf{F} = \vec{\nabla} x^2 y^2$), therefore vanishes on every closed curve, i.e. no integration required here.

Curl and Divergence (cont)

- Example: Evaluate $\oint_C \frac{x^2y^2}{\sqrt{x^2+y^2}}ds$, where *C is a circle of radius* $\sqrt{2}$*.*
- **Solution** 1:

$$\oint_C \frac{x^2y^2}{\sqrt{x^2+y^2}}ds = \int_0^{2\pi} \frac{2\cos^2 t \cdot 2\sin^2 t}{\cancel{\sqrt{2\cos^2 t+2\sin^2 t}}} \cancel{\left|\vec{r}\,'(t)\right|}dt = \int_0^{2\pi} 4\cos^2 t \sin^2 t dt = \int_0^{2\pi} \sin^2 2t dt = \left(\tfrac{t}{2} - \tfrac{1}{8}\sin 4t\right)_0^{2\pi} = \pi$$

- **Solution** 2: Notice that normal to *C* is $\mathbf{n}(t) = \frac{\langle x,y\rangle}{\left|\langle x,y\rangle\right|}$ and $\frac{x^2y^2}{\sqrt{x^2+y^2}} = \underbrace{\frac{1}{2}\left\langle xy^2, x^2y\right\rangle}_{=\mathbf{F}} \cdot \underbrace{\frac{\langle x,y\rangle}{\left|\langle x,y\rangle\right|}}_{\mathbf{n}}$

thus

$$\oint_C \mathbf{F}\cdot\mathbf{n}ds = \iint_D \operatorname{div}\mathbf{F}(x,y)dA = \frac{1}{2}\iint_D y^2 + x^2 dA = \frac{1}{2}\int_0^{2\pi}\int_0^{\sqrt{2}} r^2 \cdot r dr d\theta = \frac{1}{2}\cdot 2\pi \cdot \left(\frac{r^4}{4}\right)_0^{\sqrt{2}} = \pi$$

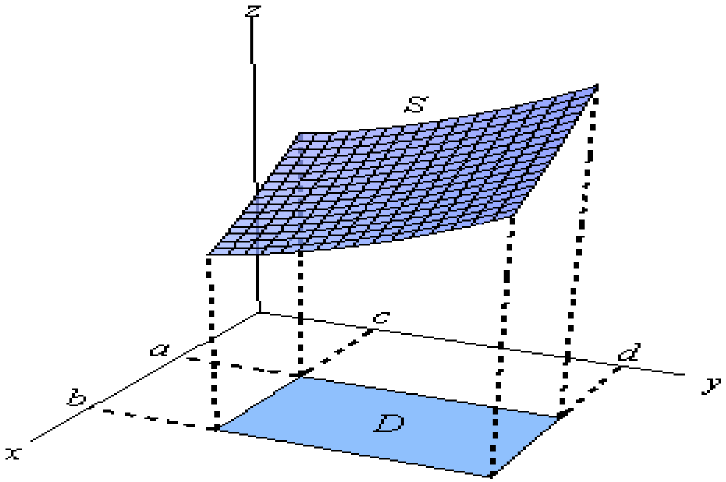
Surface Integral

- **Definition**: Let S be surface $\vec{r}(u,v)=\langle x(u,v), y(u,v), z(u,v)\rangle, (u,v)\in D$. Let P_{ij}^* be a sample point on a patch S_{ij} which area is given by $\Delta S_{ij} = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$.
  - S_{ij} is defined by $[u_{i-1}, u_i]\times[v_{j-1}, v_j]$, consequently

$$P_{ij}^* = \left\langle x\left(u_i^*, v_j^*\right), y\left(u_i^*, v_j^*\right), z\left(u_i^*, v_j^*\right)\right\rangle, u_i^* \in [u_{i-1}, u_1], v_j^* \in [v_{j-1}, v_j]$$

The surface integral is given by

$$\oiint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(P_{ij}^*\right)\Delta S_{ij}$$



Surface Integral(cont)

- **Theorem**: $\oiint_S f(x,y,z)\,dS = \iint_D f(r(u,v))\left|r_u \times r_v\right| dA$
- Note that the area of the surface *S* is $A(S) = \oiint_S 1\,dS = \iint_D \left|r_u \times r_v\right| dA$
- **Reminder**: $\oint_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\left|\vec{r}\,'(t)\right| dt$
- Note: The relationship of surface integral and surface area are analogical to the relationship between the line integral and arclength.
- Example:

$$\iint_{\partial D: z = g(x,y)} f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{(g_x)^2 + (g_y)^2 + 1}\,dA$$

Surface Integral(cont)

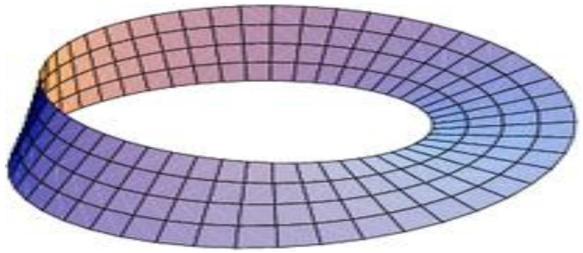
- **Example**: Let *S* be a unit sphere. The parametric representation is given by $\vec{r}\left(\varphi,\theta\right)=\left\langle \sin\varphi\cos\theta,\sin\varphi\sin\theta,\cos\varphi\right\rangle, 0\le\varphi\le\pi, 0\le\theta\le 2\pi$

$$\begin{aligned}\left|\vec{r}_\varphi\times\vec{r}_\theta\right| &=\left|\left\langle \cos\phi\cos\theta,\cos\phi\sin\theta,-\sin\phi\right\rangle\times\left\langle -\sin\phi\sin\theta,\sin\phi\cos\theta,0\right\rangle\right|\\ &=\left|\left\langle \sin^2\phi\cos\theta,\sin^2\phi\sin\theta,\cos\phi\sin\phi\right\rangle\right|=\sin\varphi\end{aligned}$$

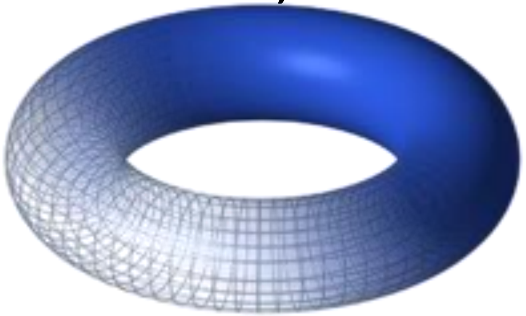
$$\begin{aligned}\oiint_S x^2 dS &= \iint_D \sin^2\varphi\cos^2\theta\left|\vec{r}_\varphi\times\vec{r}_\theta\right|dA=\iint_D \sin^3\varphi\cos^2\theta dA\\ &=\int_0^{2\pi}\cos^2\theta d\theta\int_0^{\pi}\sin^3\varphi d\varphi=\int_0^{2\pi}\frac{1+\cos 2\theta}{2}d\theta\int_0^{\pi}\left(\sin\varphi-\sin\varphi\cos^2\varphi\right)\varphi d\varphi\\ &=\frac{1}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)_0^{2\pi}\left(-\cos\varphi+\frac{1}{3}\cos^3\varphi\right)_0^{\pi}=\frac{4\pi}{3}\end{aligned}$$

Surface Integral (cont)

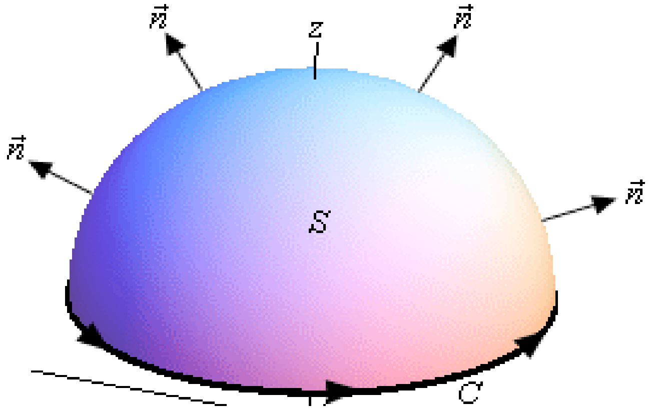
- **Definition**: A two-sided surface *S* is called oriented if the unit normal vector **n** is defined at every point (except the boundary points). The orientation is chosen by direction (positive or negative) of the unit normal, in other words by choosing the side.



- **Example**: A Mobius strip is one-sided surface, therefore non orientable.



- A torus is two-sided, so it can be oriented inward or outward.
- **Definition**: A closed surface is a boundary of solid region. A closed surface is considered positive oriented if the unit normal points outward.



Surface Integral (cont)

- **Definition**: Flux of **F** across surface S is defined by $\oiint_S \mathbf{F}\cdot d\mathbf{S} = \oiint_S \mathbf{F}\cdot\mathbf{n}\,dS$
- Notice the difference between $d\mathbf{S}$ and dS, and the similarity with $d\mathbf{r}$.
- Evaluation:

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \oiint_S \mathbf{F}\cdot\mathbf{n}\,dS = \oiint_S \mathbf{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{\left|\vec{r}_u\times\vec{r}_v\right|}\,dS$$

$$= \iint_D \left(\mathbf{F}\left(\vec{r}\left(u,v\right)\right)\cdot\frac{\vec{r}_u\times\vec{r}_v}{\left|\vec{r}_u\times\vec{r}_v\right|}\right)\left|\vec{r}_u\times\vec{r}_v\right|dA = \iint_D \mathbf{F}\cdot\left(\vec{r}_u\times\vec{r}_v\right)dA$$

Surface Integral (cont)

- $\oiint_{\partial D: z=g(x,y)} \mathbf{F} \cdot d\mathbf{S} = \iint_D \langle P, Q, R \rangle \cdot \langle -g_x, -g_y, 1 \rangle dA = \iint_D -Pg_x - Qg_y + R dA$
- Rate of flow of a fluid with density ρ and velocity field $\vec{v}$: $\oiint_S \rho \mathbf{v} \cdot \mathbf{n} dS$
- Electric flux of an electric field **E** through the surface *S*: $\oiint_S \mathbf{E} \cdot d\mathbf{S}$.
- **Gauss's Law**: a net charge enclosed by a closed surface *S:* $Q = \varepsilon_0 \oiint_S \mathbf{E} \cdot d\mathbf{S}$ *where* ε_0 is permittivity of free space.
- Let K be a conductivity constant of a substance and let u(*x*,*y*,*z*) denote temperature of a body. The of heat flow is given by $-K\nabla u$ and the rate of heat flow by $-K \oiint_S \nabla u \cdot d\mathbf{S}$.

Surface Integral(cont)

- **Example**: Let **F=**<*x,y,z*> *and* $\vec{r}\left(\varphi,\theta\right)=\left\langle \sin\varphi\cos\theta,\sin\varphi\sin\theta,\cos\varphi\right\rangle, 0\le\varphi\le\pi, 0\le\theta\le 2\pi$

$$
\begin{aligned}
\vec{r}_{\varphi}\times\vec{r}_{\theta} &= \left\langle \cos\phi\cos\theta,\cos\phi\sin\theta,-\sin\phi\right\rangle\times\left\langle -\sin\phi\sin\theta,\sin\phi\cos\theta,0\right\rangle \\
&= \left\langle \sin^{2}\phi\cos\theta,\sin^{2}\phi\sin\theta,\cos\phi\sin\phi\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}\left(\vec{r}\left(\varphi,\theta\right)\right)\cdot\left(\vec{r}_{\varphi}\times\vec{r}_{\theta}\right) &= \vec{r}\cdot\left(\vec{r}_{\varphi}\times\vec{r}_{\theta}\right) \\
&= \sin^{2}\phi\cos\phi\cos^{2}\theta+\sin^{2}\phi\cos\phi\sin^{2}\theta-\sin^{2}\phi\cos\phi \\
&= \sin\varphi
\end{aligned}
$$

$$
\oiint_{C}\mathbf{F}\cdot d\mathbf{S}=\iint_{D}\mathbf{F}\left(\vec{r}\left(\varphi,\theta\right)\right)\cdot\left(\vec{r}_{\varphi}\times\vec{r}_{\theta}\right)dA=\iint_{D}\sin\varphi\, dA=\int_{0}^{2\pi}\int_{0}^{\pi}\sin\varphi\, d\varphi\, d\theta=-2\pi\left(\cos\varphi\right)_{0}^{\pi}=4\pi
$$