Vector Calculus

part IV

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Stokes' Theorem

- Let *S* be an oriented piecewise-smooth surface that is bounded by a simple closed piecewise-smooth boundary curve *C* with positive orientation. Let **F** be a vector field whole components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains *S*. Then,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \oiint_{S} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Stokes' Theorem(cont)

- Stokes' Theorem says that the line integral around the boundary curve of *S* of the tangential component of **F** is equal to the surface integral over of the normal component of the curl of **F**.
- One of the important uses of Stoke's Theorem is in calculating surface integrals over "non convenient" surface using surface integral over more convenient surface with the same boundary:

$$\oiint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{S_1 = \partial S = S_2} \mathbf{F} \cdot d\mathbf{r} = \oiint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Stokes' Theorem(cont)

- One see Stokes' Theorem as a sort of higher dimensional version of Green's theorem. Really, if *S* is flat and lies in xy plane, then **n=k** and therefore

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \oiint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oiint_S \left(\operatorname{curl} \mathbf{F}\right) \cdot \mathbf{k}\, dS$$

which is a vector form of Green's theorem.

- Thus, Green's theorem is a private case of Stokes Theorem.

Stokes' Theorem(cont)

- **Proof** (of the light version): We restrict our proof only for the case of S given as z=g(x,y).

$$\oint_{\partial S} \mathbf{F}\cdot d\mathbf{r} = \int_a^b P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \int_a^b P dx + Q dy + R\left(z_x dx + z_y dy\right)$$

$$= \int_a^b \left(P + R z_x\right) dx + \left(Q + R z_y\right) dy \underset{Green's}{=} \iint_D \frac{\partial}{\partial x}\left(Q + R z_y\right) - \frac{\partial}{\partial y}\left(P + R z_x\right) dA$$

$$\begin{pmatrix} f(x) = g(x, y(x)) \Rightarrow \\ f' = g_x + g_y y' \end{pmatrix} = \iint_D \left(Q_x + Q_z z_x + \left(R_x + \cancel{R_z z_x}\right) z_y + \cancel{R z_{yx}}\right) - \left(P_y + P_z z_y + \left(R_y + \cancel{R_z z_y}\right) z_x + \cancel{R z_{xy}}\right) dA$$

$$= \iint_D -\left(R_y - Q_z\right) z_x - \left(P_z - R_x\right) z_y + \left(Q_x - P_y\right) dA$$

$$= \iint_D \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \cdot \left\langle -z_x, -z_y, 1 \right\rangle dA = \iint_D \operatorname{curl} \mathbf{F} \cdot \left(\vec{r}_u \times \vec{r}_v\right) dA = \oiint_{\partial D: z = g(x,y)} \mathbf{F} \cdot d\mathbf{S}$$

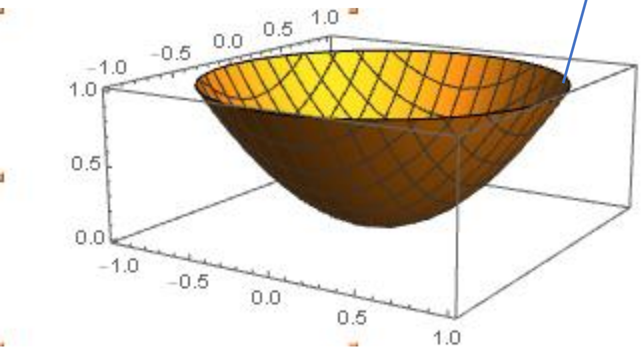
Stokes' Theorem(cont)

- **Example**: Verify Stokes' Theorem for $\mathbf{F} = \langle yz, xz, xy \rangle$ over S: $z = x^2 + y^2 \le 1$

$$\oint_{\langle \cos t, \sin t, 1 \rangle} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\left(\langle \cos t, \sin t, 1 \rangle\right) \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} \langle \sin t, \cos t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} -\sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} \cos 2t \, dt = \left. \frac{\sin 2t}{2} \right|_0^{2\pi} = 0$$



- From the other side we have, $\mathbf{F} = \nabla xyz$, therefore $\oiint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \oiint_S 0 \cdot d\mathbf{S} = 0$

Stokes' Theorem(cont)

- **Example**: Evaluate

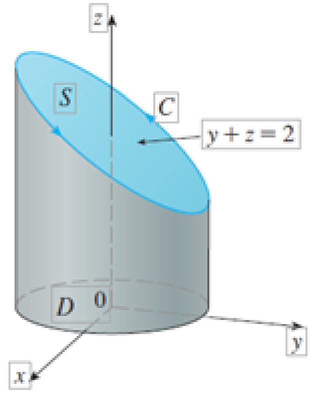
$$I = \oint_{\langle \cos t, \sin t, 2\rangle} \left(e^{-x^2/2} - yz\right)dx + \left(e^{-y^2/2} + xz + 2x\right)dy + \left(e^{-z^2/2} + 5\right)dz$$

- **Solution**: It is clear that a direct evaluation of the line integral is awkward. Therefore, denote $\mathbf{F} = \left\langle e^{-x^2/2} - yz, e^{-y^2/2} + xz + 2x, e^{-z^2/2} + 5 \right\rangle$, and use Stokes' Theorem. We also need $\operatorname{curl}\mathbf{F} = \langle x, -y, 2+2z\rangle$. Finally,

$$I \underset{Stokes}{=} \oiint_{disc} \langle x, -y, 2+2z\rangle \cdot \mathbf{n}dS \underset{\substack{\mathbf{n}=\mathbf{k} \\ z=2}}{=} 6\oiint_{disc} dA = 6A(disc) = 6\pi$$

Stokes' Theorem(cont)

- Example: Let $\mathbf{F} = \left\langle -y^2, x, z^2 \right\rangle$ and C be an intersection between cylinder $x^2 + y^2 = 1$ and $y + z = 2$. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.



$$\vec{r}(t) = \langle \cos t, \sin t, 2 - \sin t \rangle$$

$$\begin{aligned} \mathbf{F}(\vec{r}(t)) \cdot d\mathbf{r} &= \left\langle -\sin^2 t, \cos t, (2 - \sin t)^2 \right\rangle \cdot \langle -\sin t, \cos t, -\cos t \rangle \\ &= \sin^3 t + \cos^2 t - (2 - \sin t)^2 \cos t \end{aligned}$$

- Thus direct integration won't be nice; therefore we try Stokes. Let the surface S be elliptical region on plane $y + z = 2$.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S:z=2-y} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{D:x^2+y^2=1} \underbrace{\langle 0, 0, 1 + 2y \rangle}_{\operatorname{curl} \mathbf{F}} \cdot \underbrace{\langle -g_x, -g_y, 1 \rangle}_{\mathbf{n}} dA = \iint_D 1 + 2y dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r \sin\theta) r dr d\theta = \int_0^{2\pi} \left(\frac{r^2}{2} + 2 \cdot \frac{r^3}{3} \sin\theta \right)_0^1 d\theta = \int_0^{2\pi} \frac{1}{2} + 2 \cdot \frac{1}{3} \sin\theta d\theta = \frac{1}{2}\theta - \frac{2}{3}\cos\theta \Big|_0^{2\pi} = \pi$$

Divergence Theorem

- Let *E* be a simple solid region and let *S* be the boundary surface of *E*, given with positive (outward) orientation. Let **F** be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

- Note the similarity with Normal Component Green's Theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F}(x, y) \, dA$$

Divergence Theorem(cont)

- **Proof**: Let **F**=<*P,Q,R*>, we want to show

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E P_x\,dV + \iiint_E Q_y\,dV + \iiint_E R_z\,dV \underset{\text{show}}{=} \oiint_S P\mathbf{i}\cdot\mathbf{n}\,dS + \oiint_S Q\mathbf{j}\cdot\mathbf{n}\,dS + \oiint_S R\mathbf{k}\cdot\mathbf{n}\,dS$$

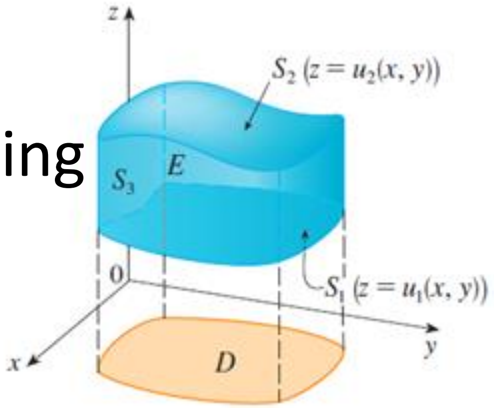
$$= \oiint_S \mathbf{F}\cdot\mathbf{n}\,dS = \oiint_S \mathbf{F}\cdot d\mathbf{S}$$

- Consider $E \underset{\text{type I}}{=} \{(x,y,z):(x,y)\in D, u_1(x,y)\le z\le u_2(x,y)\}$, then

$$\iiint_E R_z\,dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} R\,dz\,dA = \iint_D R(x,y,u_2(x,y)) - R(x,y,u_1(x,y))\,dA$$

$$= \oiint_{S_1} R\mathbf{k}\cdot\mathbf{n}\,dS + \overset{\text{either } \mathbf{k}\cdot\mathbf{n}=0 \text{ or } S_3=\phi}{\cancel{\oiint_{S_3} R\mathbf{k}\cdot\mathbf{n}\,dS}} - \oiint_{S_2} R\mathbf{k}\cdot\mathbf{n}\,dS = \oiint_S R\mathbf{k}\cdot\mathbf{n}\,dS$$

- $\iiint_E P_x\,dV = \oiint_S P\mathbf{i}\cdot\mathbf{n}\,dS$, $\iiint_E Q_y\,dV = \oiint_S Q\mathbf{j}\cdot\mathbf{n}\,dS$ are proved in a similar manner using the expressions for *E* as a type II or type III region, respectively.

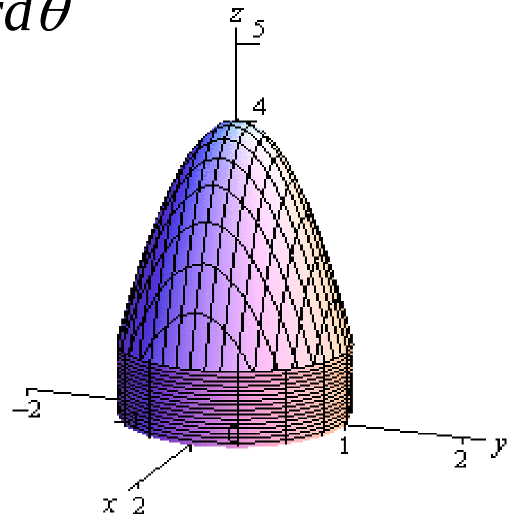


Divergence Theorem(cont)

- **Example**: Let $\mathbf{F} = \left\langle xy, -\tfrac{1}{2}y^2, z \right\rangle$ and *S* be defined by $z = 4 - 3x^2 - 3y^2, 1 \le z \le 4$ on top, $x^2 + y^2 = 1, 0 \le z \le 1$ on sides and $z = 0$ at the bottom.

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E \cancel{y - y} + 1 dV = \int_0^{2\pi} \int_0^1 \int_0^{4-3r^2} r dz dr d\theta = \int_0^{2\pi} \int_0^1 rz\Big|_0^{4-3r^2} dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 r\left(4 - 3r^2\right) dr d\theta = 2\pi \left(2r^2 - \frac{3}{4} r^4\right)_0^1 = \frac{5}{2}\pi$$



Divergence Theorem(cont)

- **Example**: Let $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$, *S* spherical solid of radius 2 in first octant.

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV = 3\iiint_E x^2 + y^2 + z^2 dV = 3\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 \rho^4 \sin\varphi d\rho d\varphi d\theta =$$

$$= 3\int_0^{\pi/2}\int_0^{\pi/2} \left.\frac{\rho^5}{5}\right|_0^2 \sin\varphi d\varphi d\theta = \frac{3 \cdot 32}{5}\frac{\pi}{2}\int_0^{\pi/2} \sin\varphi d\varphi = -\frac{48}{5}\cos\varphi\Big|_0^{\pi/2} = \frac{48}{5}$$

- **Example**: Let $\mathbf{F} = \langle 3y\cos z, x^2 e^z, x\sin y \rangle$

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E 0 dV = 0$$

Decision Tree