Handout 4

Definition: A 2x2 matrix is a mathematical entity of the following form $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Similarly, a 3x3 matrix has a form $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

Definition: Every matrix is associated with a special number called **determinant**. The determinant of 2x2 matrix is given by $\det A=|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$. The determinant of 3x3 matrix is given in terms of determinants of 2x2 matrices (called minors, and denoted M_{ij}) as following

$$\det A=|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}|M_{11}|-a_{12}|M_{12}|+a_{13}|M_{13}|=$$

$$=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}=$$

$$=a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{12}(a_{21}a_{33}-a_{23}a_{31})+a_{13}(a_{21}a_{32}-a_{22}a_{31})$$

Hint: The Minor M_{ij} obtained by erasing the ith column and jth row of the matrix A

$$M_{11}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix};\quad M_{12}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix};\quad M_{13}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Definition: The cross product between vectors $\vec{a}=\langle a_1,a_2,a_3\rangle$ and $\vec{b}=\langle b_1,b_2,b_3\rangle$ is given by

$$\vec{a}\times\vec{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=(a_2b_3-a_3b_2)\mathbf{i}-(a_1b_3-a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}=$$

$$=\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\rangle$$

Notes:

- cross product is not commutative, i.e. $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$
- $\vec{a}\times\vec{a}=\langle a,b,c\rangle\times\langle a,b,c\rangle=\langle a_2a_3-a_3a_2,a_3a_1-a_1a_3,a_1a_2-a_2a_1\rangle=\vec{0}$
- if $b=k\vec{a}$ then $\vec{a}\times\vec{b}=0$
- $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\neq\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

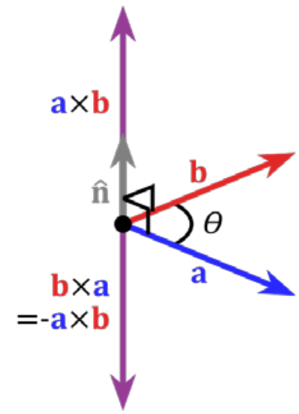
Properties of Cross Product

1) $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$

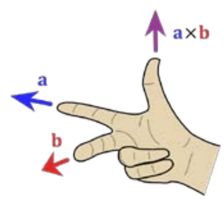
2) $(k\mathbf{a})\times\mathbf{b}=k(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times(k\mathbf{b})$

3) $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$

4) $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$



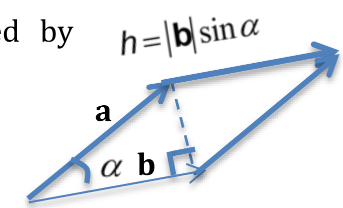
Definition: Geometrical definition of cross product is given by $\vec{a}\times\vec{b}=\left(\|\vec{a}\|\|\vec{b}\|\sin\alpha\right)\mathbf{n}$ where α is the angle between the vectors $\vec{a}$ and $\vec{b}$ and $\boldsymbol{n}$ is a unit vector perpendicular\orthogonal to both vectors $\vec{a}$ and $\vec{b}$.



Corollary: $\vec{a}\times\vec{b}$ orthogonal to both vectors $\vec{a}$ and $\vec{b}$.

Corollary: Non zero vectors $\vec{a}$ and $\vec{b}$ are parallel iff $\vec{a}\times\vec{b}=0$.

Theorem: $\|\vec{a}\times\vec{b}\|=\|\vec{a}\|\|\vec{b}\||\sin\alpha|$ is area of parallelogram determined by vectors $\vec{a}$ and $\vec{b}$.



Definition: An angle between 2 vectors defined $\|\sin\alpha\|=\dfrac{\|\vec{a}\times\vec{b}\|}{\|\vec{a}\|\|\vec{b}\|}$. Thus

$$0\le\alpha\le\frac{\pi}{2}:\alpha=\arcsin\frac{\|\vec{a}\times\vec{b}\|}{\|\vec{a}\|\|\vec{b}\|}\qquad -\frac{\pi}{2}\le\alpha\le 0:\alpha=-\arcsin\frac{\|\vec{a}\times\vec{b}\|}{\|\vec{a}\|\|\vec{b}\|}$$

Definition: **Scalar Triple product** of vectors $\vec{a},\vec{b},\vec{c}$ is defined to be $\vec{a}\cdot(\vec{b}\times\vec{c})$.

If we think about parallelepiped defined by vectors $\vec{a},\vec{b},\vec{c}$ with with a base of parallelogram defined by $\vec{b},\vec{c}$. The height of the parallelogram is $\boldsymbol{h}=|\vec{a}\cos\theta|$ and the area of the base is $A=\|\vec{b}\times\vec{c}\|$.

Definition: The volume parallelepiped defined by vectors $\vec{a},\vec{b},\vec{c}$ is given by:

1) $V=hA=\|\vec{a}\||\cos\theta|\|\vec{b}\times\vec{c}\|=\|\vec{a}\|\|\vec{b}\times\vec{c}\||\cos\theta|=\vec{a}\cdot(\vec{b}\times\vec{c})$.

2) $\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{c}\cdot(\vec{a}\times\vec{b})=(\vec{a}\times\vec{b})\cdot\vec{c}$

Definition: The **vector triple product is** given by $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$